# A Robust Model of Bubbles
# with Multidimensional Uncertainty

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**Abstract**

Boom-bust episodes in asset markets have often been interpreted by observers as speculative frenzies where asymmetrically informed investors buy overvalued assets hoping to sell to a *greater fool* before the crash. While intuitively appealing, this notion of speculative bubbles has been difficult to reconcile with standard economic theory. Existing models of speculation have been criticized on the basis that they assume irrationality, that prices are somewhat unresponsiveness to sales, or that they depend on fragile, knife-edge restrictions. To address these issues, I construct a rational version of Abreu and Brunnermeier (2003), where agents invest growing endowments into an asset, fueling appreciation and eventual overvaluation. Riding bubbles is optimal as long as the price grows quickly and there is a probability of exiting before the crash. This probability increases with the amount of noise in the economy, as random price flucutations make it difficult for agents to infer sales from the price.

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# 1 Introduction

Over the last two decades, a series of dramatic boom-bust episodes in global asset markets have led many economists to give increasing consideration to theories of asset price bubbles and to doubt the long dominant efficient market hypothesis.

Asset price bubbles are often called speculative bubbles, a term that already suggests a conceptualization of bubbles in which market timing plays a crucial role, with investors buying overvalued assets hoping to sell to a *greater fool* before the crash. This idea is not new. For instance, Kindleberger and Aliber (2005) describe numerous historical boom-bust episodes that were seen by contemporaneous observers as speculative in this sense. But despite its intuitive appeal, the notion of a speculative bubble has traditionally been difficult to reconcile with standard economic theory. As Tirole (1982) and Milgrom and Stokey (1982) show, in a wide range of environments with finite numbers of rational agents, bubbles are inconsistent with rational expectations equilibrium, even under asymmetric information. Roughly, the intuition behind these results is that trade in overvalued assets is a zero-sum game, and that, on average, agents cannot rationally expect to have better-than-average information.

Some approaches that have been taken in order to circumvent these impossibility results include, among others, introducing some form of irrationality, having heterogenous priors/marginal utilites, and assuming an infinite number of overlapping generations. For instance, Harrison and Kreps (1978) and Scheinkman and Xiong (2003) consider agents who are ‘overconfident’ in the sense that they consider their own information to be superior to that of others, and fail to fully adjust their beliefs as they observe what others believe. In Abreu and Brunnermeier (2003), there are rational agents who ride the bubble—and make profits with a certain probability—along with behavioral agents who fuel bubble growth and who are certain to suffer losses in the crash. In Allen et al. (1993) and Conlon (2004), agents are rational, but have either heterogenous priors, or heterogenous state-contingent marginal utilities that may give rise to gains from trade. This approach generates speculative bubbles, but has the drawback of relying on fragile, knife-edge parameter restrictions. Another strand of literature (Caballero and Krishnamurthy (2006), Fahri and Tirole (2009), and others) builds on Tirole’s (1985) work on rational bubbles in overlapping generations models. In these models, bubbles improve efficiency by helping overcome a shortage of stores of value, much like money in Samuelson (1958). However, the bubbles arising in these models have perhaps less of a speculative flavor, as trades are driven by the lifecycle rather than by beliefs, and the focus is often on steady states with slow-growing bubbles.[1]

The objective of this paper is to contribute to the theory of speculation by developing a model that captures the idea of a *greater fool's* bubble but avoids the main critiques of previous models. To this end, I build a version of Abreu and Brunnermeier (2003)—henceforth referred to as AB—in which all agents are rational and prices reflect supply and demand at all times. The model inherits from AB the property of being robust to small changes in parameters, and is therefore not subject to the fragility critique of Allen et al. (1993) and Conlon (2004).

Following AB, I assume that rational agents hold a rapidly appreciating asset. For some time, rapid price growth is justified by fundamentals, but a bubble arises because the price continues to rise past the point where fundamental gains have been priced in. Asymmetric information is introduced in the environment in such a way that rational agents remain invested in the asset even after learning that is has become overvalued. At different times, different agents observe private signals revealing that a bubble has started to inflate. They do not know when others observe the signal, but they know that, in equilibrium, those who observe the signal relatively early can ride the bubble, sell before the crash and make profits. If the likelihood of being an 'early-signal' agent and the price growth rate are high enough, investing in overvalued assets is optimal. To embed these ideas into a rational model, I depart from AB in the following ways. In AB, bubble growth is fueled by behavioral agents who invest growing amounts into the risky asset and are doomed to 'get caught' in the crash. Instead of behavioral agents, I assume that rational agents are entitled to growing endowments, which they invest in the bubble as long as they do not expect an imminent crash. Importantly, these endowments cannot be pledged as collateral, i.e., agents cannot borrow against their time-$t$ endowment at some earlier date $s < t$. Besides the endowment, another new ingredient in the model is a preference shock, which forces a fraction $\theta_t$ of agents to sell for reasons—such as life events or liquidity needs—unrelated to price expectations. This ensures that a positive mass of shares is sold every period, even if nobody expects an imminent crash.[2] Preference shocks also serve another function, adding noise

[1] Further approaches to modeling bubbles, which have also been influential, have focused on issues such as agency problems (Allen and Gorton (1993), Allen and Gale (2000), Barlevy (2008)), solvency constraints (Kocherlakota (2008)), and others. For an survey, see Brunnermeier (2001).
[2] The preference shock shall not be thought of as a substitute for irrational agents, since it does not force agents to stay in the market during the crash. On the contrary, it forces some agents to sell before they otherwise would.

into the economy. The fact that $\theta_t$ is subject to random variability makes prices noisy. In turn, noisy prices 'hide' sales, as late-signal agents cannot distinguish whether a price slowdown is due to early-signal agents' sales or just a high realization of $\theta_t$. Thus, the likelihood that a given agent will be able to sell before the crash increases with the degree to which $\theta_t$ is variable. The addition of noise makes it possible for prices to reflect selling pressure at all times without being fully revealing. This helps overcome another critique of the AB model, where, in the absence of noise, it is assumed that during the (nontrivial) length of time when early-signal agents are gradually leaving the market, the price continues to grow as if nobody was selling.[3]

To solve the model, I first consider the case where there is so little noise that as soon as one type sells (a type includes those who observed the overvaluation signal in a given period), all uncertainty is revealed, triggering a crash in the next period. I derive a parameter restriction such that, in this effectively noiseless environment, agents sell immediately upon observing the signal. Maintaining this restriction, which is an upper bound on the growth rate of the bubble, I increase the amount of noise so that it can conceal sales of one type, but not more. Prices (relative to trend) then fall into one of three categories. High prices reveal with certainty that nobody has sold, medium prices reveal that sales may or may not have begun, and low prices reveal with certainty that sales have begun, thereby triggering the crash. This noise specification, coupled with Markov strategies, is simple and analytically tractable, in the case where the number of types is large. The strategies I consider are Markovian in the sense that agents' sell-or-wait choices depend only on how much time has elapsed since observing the signal and on whether the last price observed was high, medium, or low. Restricting attention to this class of strategies, I show that there is a region in the parameter space where bubbles of arbitrary length arise. Finally, I relax the assumption—made in the basic analysis for simplicity—that agents cannot reenter the market after selling, and show that, although some of the basic-analysis equilibria vanish, the overall picture remains unchanged, and bubbles with Markovian strategies still arise.

In sum, I develop a model of speculative bubbles based on AB, but with fully rational agents and prices that are always market clearing. By avoiding some critiques of previous literature, this paper makes models of speculative bubbles more compatible with standard theory, and hence potentially more useful as tools for policy analysis.

[3] The idea of adding noise to prevent full information revelation is already suggested by AB. In a different context, Allen et al. (1993) and Conlon (2004) also speculate that adding noise would make their models more robust.

The paper is organized as follows. In sections 2 and 3, respectively, I describe the model and define equilibrium. In section 4, I illustrate how bubbles arise in the basic analysis. In section 5, I present an extension and I conclude in section 6.

# 2 The Model

## 2.1 The Environment

Time is discrete and infinite with periods labeled $t = \ldots, -1, 0, 1, \ldots.$ There are two assets, a risk-free asset with exogenous gross return $R > 1,$ and a risky asset. The supply of the risky asset is fixed at $1,$ and its price at time $t$ is $p_t$ units of the risk-free asset. At any time, the risk-free asset can be turned into consumption at a one to one rate.

While $t \leq 0,$ the risky asset's fundamental value $f_t$ and the price $p_t$ are equal and given, in expected value, by $\alpha R^t,$ where $\alpha > 0.$ Starting at $t = 1,$ fundamental shocks cause $f_t$ to grow, on average, at the faster rate $G > R.$ Both $f_t$ and $p_t$ grow on average at the faster rate $G$ until $t = t_0 - 1.$ But starting at time $t_0 \geq 1,$ the average $f_t / f_{t-1}$ falls back to $R,$ and if $p_t$ continues to grow faster than $R,$ a bubble arises.[4] The bubble inflates until period $T \geq t_0$ and bursts in period $T + 1,$ at which point equality between price and fundamental value is restored. Thus, as in AB, bubbles arise as markets overreact to developments that are at first fundamental in nature.[5] The first period of overvaluation $t_0$ is geometrically distributed with probability function $\varphi$ given by

$$\varphi(t_0) = \left(e^{\lambda} - 1\right) e^{-\lambda t_0} \qquad \text{for all } t_0 = 1, 2, \ldots, \tag{1}$$

where $\lambda > 0.$ The expected value of $t_0$ is given by $1/(1 - e^{-\lambda}).$ Also, the greater $\lambda,$ the greater the probability of low values of $t_0$ relative to high values.

There is a unit mass of rational agents, indexed by $i \in [0,1].$ They do not observe $t_0$ perfectly. Instead, every period from $t_0$ to $t_0 + N - 1,$ a mass $1/N$ of them observe a signal

[4] As in AB, the increase in fundamental value does not stem from a rise in current dividends (set equal to zero for convenience), but from improving prospects about the future dividends that the risky asset may yield.

[5] According to Kindleberger and Aliber (2005), bubbles typically follow *displacements*, i.e., major fundamental events that cause sizable shifts in prices. Price movements that are justified by fundamentals for some time may turn into bubbles if markets overshoot. In keeping with this idea, AB mention episodes in stock markets after the arrival of new technologies (e.g., the Internet in the 1990s, the radio in the 1920s) as examples of bubbles. In these cases, prices rose dramatically, then crashed, and finally stabilized at a level higher than before the fundamental change but below the peak. In Doblas-Madrid (2010), I argue that the idea of a bubble as a temporary overreaction to fundamental events can also help to explain exchange rate overshooting in a series of currency crisis episodes.

revealing that the risky asset is overvalued, i.e., that $f_t$ is no longer growing at the rate $G$. Signals give rise to $N$ types, $n = t_0, \ldots, t_0 + N - 1.$ The function $\nu : [0,1] \to \{t_0, \ldots, t_0 + N - 1\}$ assing a type to each agent. Thus, $\nu(i) = n$ denotes that agent $i$ is of type $n$, or, in other words, that agent $i$ observes the signal at time $n$. As in AB, agents observe $n$, but not $t_0$. If an agent observes her signal at time $n$, she knows that $t_0$ may have been as early as $n-(N-1)$, or as late as $n$. (Except for the special case with $t_0 < N$, where types with $n < N$ know that $t_0$ must be greater than $n-(N-1)$, since $n-(N-1) \le 0$.) Conditional on $n$, the distribution of $t_0$ becomes

$$\varphi(t_0 \mid n) = \begin{cases} \dfrac{e^{-\lambda t_0}}{e^{-\lambda(\max\{1, n-(N-1)\})} + \cdots + e^{-\lambda n}} & \text{if } \max\{1, n-(N-1)\} \le t_0 \le n \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In words, sequential arrival of signals places agents along a line, but agents are uncertain about their relative order in the line. This plays a key role in generating bubbles, as all agents—even those late in the line—assign positive probability to the event that they could be early in the line.

Figure 1 summarizes our assumptions thus far. The boom starting at $t=1$ is fundamental in nature at first, but turns into a bubble at the imperfectly observed time $t = t_0$. Signals arrive at $t = t_0, \ldots, t_0 + N - 1$, and bubble duration $T - t_0$ will be endogenously determined in equilibrium.

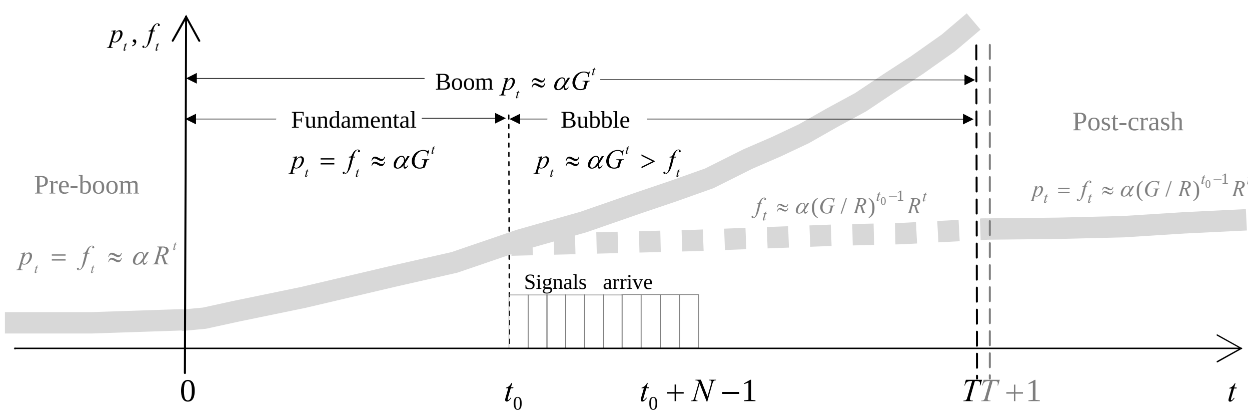


**Figure 1** — Timeline of events.

Preferences are characterized by risk neutrality and preference shocks à la Diamond and Dybvig (1983), which may force agents to liquidate assets and consume. At time $t$, a randomly chosen mass $\theta_t \in (0,1)$ of agents are hit by a shock that sets their discount factor $\delta_{i,t}$ equal to zero. The remaining mass $1-\theta_t$ have $\delta_{i,t} = 1/R.$ Agent *i*'s expected utility is defined as

$$E_{i.t}\left[U\left(\{c_{i.\tau}\}_{\tau=t}^{\infty}\right)\right] = E_{i,t}\left[c_{i,t} + \sum_{\tau=t+1}^{\infty}\left(\prod_{s=t}^{\tau-1}\delta_{i,s}\right)c_{i,\tau}\right], \tag{3}$$

where $c_{i,t}$ denotes agent *i*'s time-*t* consumption, $U$ denotes utility, and $E_{i,t}$ expectation given information available to agent $i$ in period $t.$ This includes whether $\delta_{i,t}$ is zero, in which case the above simplifies to $E_{i,t}\left[c_{i,t}\right].$ Preference shocks are i.i.d., and thus, the probability that $\delta_{i,t} = 0$ does not depend on past values $\ldots, \delta_{i,t-2}, \delta_{i,t-1}.$ Shocks are also type-independent, which implies that at all times, within any type, the fraction of agents whose discount factor equals zero is $\theta_t.$

Since $\theta_t$ is unobservable, agent $i$ knows whether she has been hit by the shock, but not how many agents have been hit. Moreover, $\theta_t$ varies over time as follows:

$$\theta_t = \overline{\theta} + \varepsilon_t,$$

where $\overline{\theta} \in (0,1)$ is a constant and $\varepsilon_t$ is an i.i.d. random variable which is uniformly distributed over $[-\overline{\varepsilon}, \overline{\varepsilon}],$ with $0 < \overline{\varepsilon} < \min\{\overline{\theta}, 1-\overline{\theta}\}.$ The term $\varepsilon_t$ serves an important function in the model by generating random price fluctuations. If $\theta_t$ was constant, as soon as the first agents sold in anticipation of the crash, the price would reveal these sales and precipitate a crash. In a noisy environment, by contrast, agents cannot distinguish whether a price deceleration is due to a high $\varepsilon_t$ or the start of the crash. It is important to note that the role of preference shocks is precisely to generate a positive and noisy amount of sales. The role of the shock is *not* to make speculation a positive sum game by making some agents get caught in the crash. The shock never forces any agents to stay in the market. On the contrary, it sometimes forces agents to sell the risky asset.

The bubble is fueled by agents who invest their endowment into the risky asset. Every period, agents receive $e_t > 0$ units of the risk-free asset. If they do not anticipate an impending crash and are not hit by the preference shock, they choose to invest it into the risky asset.

Endowments cannot be capitalized, i.e., agents receive $e_t$ at time $t$ and cannot be pledge $e_t$ in order to borrow at earlier dates $s < t.$ After time 0, endowment growth accelerates as follows:[6]

$$e_t = \begin{cases} R^t & \text{if } t \leq 0 \\ G^t & \text{if } t > 0. \end{cases} \tag{4}$$

Three remarks are in order. First, the assumption that $e_t$ grows at the rate $G$ forever shall not be interpreted literally. In the long run, endowment growth must eventually slow down. However, limits on endowment growth are not explicitly modeled because the focus of the paper is on endogenous crashes, where agents' sales will burst the bubble before it starts to decelerate for exogenous reasons.[7] Second, agents' inability to borrow against future endowments is a crucial difference between this model and Tirole (1982), where agents' borrowing ability is unlimited. Here, as in AB, agents are wealth constraint and an inflow of new funds fuels growth. However, unlike in AB, in this model new money is not 'dumb money'. Agents invest endowments into the boom only if doing so is optimal. Thus, funds invested into the bubble relatively late are not any more likely than funds invested early to get caught in the crash. The third remark is that an alternative specification with a constant $\theta_t$ and a noisy aggregate endowment would also generate price fluctuations, but it would not generate fluctuations in trading volume.

The within-period timing of shocks and actions is as follows. Agent $i$ starts period $t$ with nonnegative holdings $b_{i,t}$ and $h_{i,t}$ of the risk-free and risky assets, respectively. The period proceeds in two steps. In Step 1, agent $i$ receives $e_t$, learns whether $\delta_{i,t}$ is zero or $1/R,$ and, if $\nu(i) = t,$ observes her signal. Also in Step 1, agent $i$—knowing $\delta_{i,t},$ $p^{t-1} = \{\ldots, p_{t-2}, p_{t-1}\},$ and if $\nu(i) \leq t,$ the signal—chooses actions $a_{i,t} = (m_{i,t}, s_{i.t}, \chi_{i,t}).$ The pair $(m_{i,t}, s_{i,t})$ captures the agent's asset market choices, while $\chi_{i,t} \in [0,1]$ captures the agent's consumption choice. (Although consumption takes place in Step 2, the decision whether to consume or not depends only on the

[6] These endowments may originate from multiple sources. One source may be labor income, which is usually only partially pledgeable. Regarding other sources of new funds to fuel the bubble, Kindleberger () points to the expansion of credit and the arrival of new investors into the market. Thus, growing endowments may reflect the loosening of credit constraints as the bubble grows and agents' net worth rises. To explain the gradual arrival of new investors, one could imagine a world with imperfect information diffusion, where time constraints limit the number of markets agents can follow. News of the boom would attract new investors, who would feed the boom even further, attracting even more agents, and so forth. This self-reinforcing cycle would accelerate growth for some time, before eventually slowing down as the fraction of agents invested in the bubble approached 1.

[7] A similar issue arises in AB, where behavioral agents are assumed able to purchase a given number of shares of the risky asset no matter how high the price becomes, but there is also an exogenous cap on bubble duration.

preference shock, which is realized in Step 1.) In the asset market—modeled à la Shapley-Shubik—agent $i$ bids $m_{i,t}$ units of the risk-free asset and offers $s_{i,t}$ shares of the risky asset for sale. Due to short sales constraints,

$$0 \leq m_{i,t} \leq b_{i,t} + e_t \tag{5}$$

and

$$0 \leq s_{i,t} \leq h_{i,t} \tag{6}$$

must hold. Agent *i* chooses $(m_{i,t}, s_{i,t})$ before knowing the price $p_t$, which will be determined in Step 2 when all bids and offers are combined.[8] Preference shocks and risk neutrality greatly simplify agent's asset market choices. Agents with $\delta_{i,t} = 0$ sell everything to consume as much as possible in Step 2, i.e., they set $(m_{i,t}, s_{i,t}) = (0, h_{i,t})$. Agents with $\delta_{i,t} = 1/R$ set $(m_{i,t}, s_{i,t}) = (0, h_{i,t})$ if they expect the risky asset's return $p_{t+1} / p_t$ to fall below $R$; they invest as much as they can into the risky asset, setting $(m_{i,t}, s_{i,t}) = (b_{i,t} + e_t, 0)$ if this expected return exceeds $R$, and are indifferent between any linear combination of these two actions in the knife-edge case. Agent $i$ will come out of the asset market with asset holdings given by

$$h_{i,t+1} = h_{i,t} + \frac{m_{i,t}}{p_t} - s_{i,t} \tag{7}$$

and

$$\tilde{b}_{i,t} = b_{i,t} + e_t - m_{i,t} + p_t s_{i,t}, \tag{8}$$

where $\tilde{b}_{i,t}$ denotes agent *i*'s within-period, or interim, risk-free asset holdings.

In Step 2, offers and bids are combined and the price is determined by market clearing

$$\int_{i \in [0,1]} h_{i,t} di = 1. \tag{9}$$

Substituting (7) into this expression and solving for the price yields

$$p_t = \frac{M_t}{S_t}, \tag{10}$$

where, for any $t$,

[8] The assumption that agents submit orders before knowing others' orders or the price is similar to Kyle (1985), and also to models a la Cournot. In the jargon of financial markets, agents are placing *market orders*, which they know will be executed, but they do not know at what price.

$$M_t \equiv \int_{i\in[0,1]} m_{i,t}di \qquad \text{and} \qquad S_t \equiv \int_{i\in[0,1]} s_{i,t}di. \tag{11}$$

Since there is always a positive measure of preference-shock induced sellers, $S_t$ is always positive and (10) is well defined. Finally, agent $i$ consumes a fraction $\chi_{i,t} \in [0,1]$ of $\tilde{b}_{i,t}$

$$c_{i,t} = \chi_{i,t}\tilde{b}_{i,t}, \tag{12}$$

and saves the rest, so that next-period's risk-free asset holdings $b_{i,t+1}$ are given by

$$b_{i,t+1} = R\left(1-\chi_{i,t}\right)\tilde{b}_{i,t}. \tag{13}$$

Figure 2 summarizes within-period timing

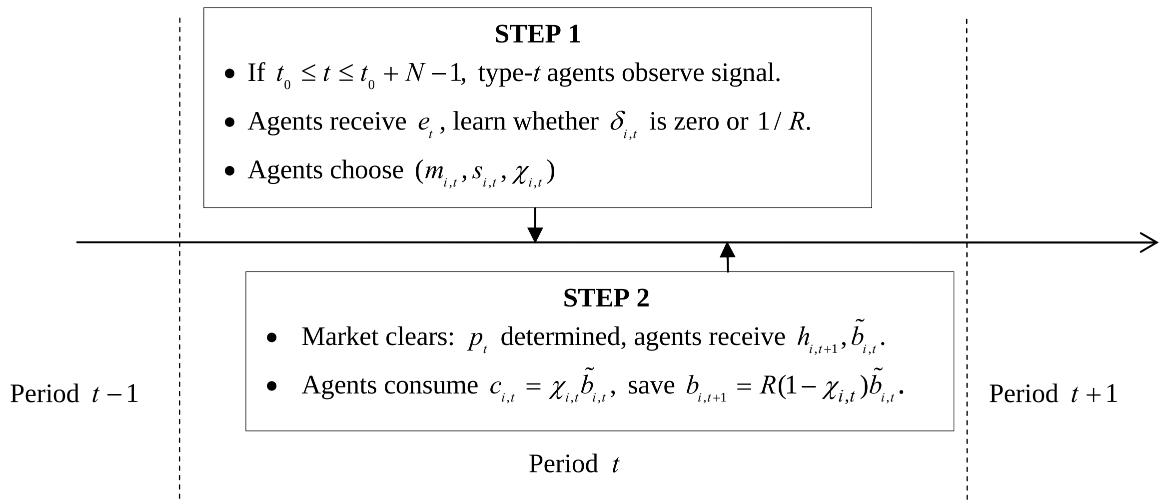


**Figure 2** — Within-period timing.

Having described market clearing, we can now fill in details about the pre-boom, boom and post-crash phases. But before proceeding, let us make two assumptions that will allow us to avoid unnecessary complications and generate price dynamics exactly as in Figure 1. The first assumption is that agents are not wealth constraint while $t \le 0$. This implies that $\alpha(G/R)^{t_0-1}R^t$ is indeed the correct post-crash expected price and fundamental value.[9] The second assumption is that $\alpha = 1/\bar{\theta} - 1$. When this holds, it is optimal for agents to hold $b_{i,t} = 0$ while $t \le 0$. In turn,

[9] If agents were wealth constraint during the pre-boom phase, the only change with respect to Figure 1 would be a delay in the start of the bubble. The post-crash price would be higher than $(G/R)^{t_0-1}R^t$ by a factor capturing the degree to which agents were wealth constraint before the boom. Thus, the fundamental part of the boom would last longer, but the duration of the bubble would not be affected.

this has the convenient implication that, at $t = 1,$ there is an increase in the price growth rate, but there is no additional ‘discrete’ jump in the price.

With these assumptions in place, consider a pre-boom period $t \leq 0,$ and let agents start with $b_{i.t} = 0$ units of the risk-free asset. Preference shocks force a mass $\theta_t$ of agents to sell their shares of the risky asset, which also amount to $S_t = \theta_t.$ Other agents use their endowments to bid for these shares. With $b_{i.t} = 0,$ we have $M_t = (1-\theta_t)e_t,$ and thus

$$p_t = \left( [\theta_t]^{-1} - 1 \right) e_t. \tag{14}$$

Since the expected $p_{t+1} / p_t$ equals $R,$ agents who are not hit by the shock find it (weakly) optimal to consume nothing and to invest their entire wealth into the risky asset, letting $b_{i,t+1} = 0.$ Since $\alpha = 1/\overline{\theta} - 1,$ the risky asset is valuable enough to store agents’ entire wealth.[10] Agents who are hit by the shock consume $c_{i,t} = \tilde{b}_{i,t} = e_t + p_t h_{i,t}$ and save nothing, i.e, let $(b_{i,t+1}, h_{i,t+1}) = (0,0).$

At $t = 1,$ endowment and price growth accelerate. For a while, the only sales are those forced by shocks, those who are not hit by the shock remain fully invested in the risky asset, and $p_t$ is given by (14) with $e_t = G^t.$ Since shocks are type-independent, in the aggregate each type holds $h_{n,t} = 1/N$ shares of the risky asset, where for all $n \in \{t_0, \ldots, t_0 + N - 1\}$ and for all $t,$

$$h_{n,t} \equiv \int_{\{i|\nu(i)=n\}} h_{i,t} di. \tag{15}$$

All $N$ types hold $h_{n,t} = 1/N$ shares until the last few periods of the boom, when some begin to sell in anticipation of the crash. When the first $z_t > 0$ types sell at $t,$ the number of shares for sale becomes $S_t = z_t / N + \theta_t (1 - z_t / N),$ where a mass $z_t / N$ of agents sell anticipating a crash and a mass $\theta_t (1 - z_t / N)$ of agents sell strictly because of preference shocks. Aggregate bids $M_t$

[10] If $\alpha$ was below $1/\overline{\theta} - 1,$ agents who are not hit by the shock would invest a fraction $\xi \equiv \alpha\overline{\theta} / (1-\overline{\theta}) \in (0,1)$ of the endowment $R^t$ in the risky asset and the rest in the risk free asset. This would imply pre-boom aggregate holdings of the risk-fee asset of $b_t = R^t (1-\xi)(1-\overline{\theta}) / (1-(1-\overline{\theta})/R).$ Once the boom began, this wealth would flow into the risky asset, causing a price jump in period 1, followed by some periods where $p_t$ would grow on average at the rate $R,$ up until the point where $b_t$ reached zero. After that point, wealth constraints would bind and $p_t$ would grow at the average rate $G > R,$ as in Figure 1. As long as $\underline{b}_0$ is small enough that wealth constraints start binding well before the bubble bursts, the assumption that $\alpha = 1/\overline{\theta} - 1$ is innocuous; it allows us to abstract from inessential price dynamics without otherwise affecting bubble duration in any way.

amount to $(1-\theta_t)(1-z_t/N)G^t$, as only agents who are not hit by the shock and are not of the exiting types wish to be long in the risky asset. Overall, thus, the price becomes

$$p_t = \left(\left[\frac{z_t}{N} + \theta_t\left(1-\frac{z_t}{N}\right)\right]^{-1} - 1\right)G^t. \tag{16}$$

After trade, $h_{n,t+1}$ equals 0 for the $z_t$ types that have sold and $(1-z_t/N)^{-1}$ for all other types. Agents hit by the shock consume the proceeds from selling the risky asset, while agents who sell without being hit by the shock store their wealth in the risk-free asset hold, i.e., they let $b_{i,t+1} > 0$. The likelihood that $p_t$ reveals the exit of these $z_t$ types depends on the relative magnitudes of $\bar{\varepsilon}$ and $z_t/N$. If $\bar{\varepsilon} < (1-\bar{\theta})z_t/(2N-z_t)$, sales will surely be revealed, as $((\bar{\theta}+\bar{\varepsilon})^{-1}-1))G^t$, the lowest possible price if $z_t = 0$ exceeds $([z_t/N + (\bar{\theta}-\bar{\varepsilon})(1-z_t/N)]^{-1}-1)G^t$, the highest possible price if $z_t$ types have sold. However, if $\bar{\varepsilon} \geq (1-\bar{\theta})z_t/(2N-z_t)$, $p_t$ may be greater or equal than $((\bar{\theta}+\bar{\varepsilon})^{-1}-1))G^t$, in which case the bubble will continue until period $t+1$.

If the bubble survives period $t$ and another $z_{t+1} \geq 0$ types sell at $t+1$, the aggregate bid becomes $M_{t+1} = (1-\theta_{t+1})(1-(z_t+z_{t+1})/N)G^{t+1}$.[11] On the selling side, $z_{t+1}/N$ sellers anticipate a crash and $\theta_{t+1}(1-(z_t+z_{t+1})/N)$ sellers are strictly shock-induced. Since risky-asset holdings across sellers average $(1-z_t/N)^{-1}$, the total mass of shares for sale equals

$$S_{t+1} = \left[1-\frac{z_t}{N}\right]^{-1}\left(\frac{z_{t+1}}{N} + \theta_{t+1}\left(1-\frac{z_t}{N}-\frac{z_{t+1}}{N}\right)\right).$$

Given this, and after rearranging terms, the equilibrium price can be written as

$$p_{t+1} = \left(1-\frac{z_t}{N}\right)\left(\frac{1-\frac{z_t}{N}}{\frac{z_{t+1}}{N}+\theta_{t+1}\left(1-\frac{z_t - z_{t+1}}{N}\right)} - 1\right)G^{t+1}. \tag{17}$$

The likelihood that $p_{t+1}/G^{t+1}$ falls below $((\bar{\theta}+\bar{\varepsilon})^{-1}-1))$ now depends on $\bar{\varepsilon}$, $z_t$ and $z_{t+1}$. If $p_{t+1}/G^{t+1}$ falls below this threshold, sales will be revealed, causing a crash. Otherwise, the bubble will last until time $t+2$ or later. Equation (17) can be generalized to accommodate sales

[11] The market clearing condition is different if $z_{t+1}$ is negative (i.e., if some types reenter the market after selling). I restrict attention to $z_{t+1} \geq 0$ because reentry does not occur in any of the equilibria presented in coming sections.

over more than two periods.[12] However, since in the equilibria analyzed later, sales burst the bubble in one or two periods, (17) lays all the groundwork necessary for our purposes.

The post-crash phase starts at time $T+1$, where $T$ is the first period in which $p_t / G^t$ falls below $((\bar{\theta}+\bar{\varepsilon})^{-1}-1))$. Assuming that $t_0$ is revealed at $T$, the expected fundamental value $\alpha\left(G/R\right)^{t_0-1} R^t$ also becomes known.[13] From time $T+1$ onward, agents who are not hit by the shock invest a decreasing fraction $\left(R/G\right)^{t-(t_0-1)}$ of their endowments in the risky asset, and the rest in the risk-free asset. This choice is weakly optimal since, throughout the post-crash phase, the expected ratio $p_{t+1}/p_t$ equals $R$. Moreover, equality between $p_t$ and $f_t$ is maintained.

# 3 Equilibrium

I next define equilibrium under the restriction—to be relaxed in Section 5—of no reentry, which means that, once an entire type has sold in anticipation of the crash, all agents of that type stay out of the market until the bubble bursts. (Note that this does not preclude agents who are forced to sell by shocks from investing their endowments in the risky asset in later periods.)

***Restriction I - No Reentry:*** *For any i and any* $t \le T$, *if* $b_{i,t} > 0$, then $h_{i,\tau} = 0$, $\forall \tau \in \{t+1, \dots, T\}$.

The equilibrium concept is Perfect Bayesian Equilibrium (PBE), given by stragegies and beliefs $\{a_i, \mu_i\}_{i\in[0,1]}$. Agent *i*'s strategy $a_i$ is a sequence $\left\{a_{i,t}\right\}_{t\in\mathbb{Z}}$, where, for all *t*, $a_{i,t}$ is a triplet $(m_{i,t}, s_{i,t}, \chi_{i,t})$. Agent *i*'s belief $\mu_{i,t}(t_0)$ is a probability distribution over values of $t_0$. Both $a_{i,t}$ and $\mu_{i,t}(t_0)$ are contingent on information available to agent $i$ in Step 1 of date $t$. This includes the discount factor $\delta_{i,t}$, past prices $p^{t-1} = \{\dots, p_{t-2}, p_{t-1}\}$, and if $\nu(i) \ge t$, the signal $\nu(i)$. Since $\delta_{i,t}$ does not inform about $t_0$, and all agents within a type observe the same prices and signal, they have the same common belief, defined as $\mu_{n,t}(t_0) \equiv \mu_{i,t}(t_0)$ for all $i$ with $\nu(i) = n$.

[12] If sales start at $t \ge 0$ and $z_t, \dots, z_{t+h} \ge 0$ types sell at times $t, \dots, t+h$, the price $p_{t+h}$ is given by (17), replacing $(\theta_{t+1}, G^{t+1}, z_{t+1})$ with $(\theta_{t+h}, G^{t+h}, z_{t+h})$ and $z_t$ with $z_t + \cdots + z_{t+h-1}$.

[13] In the equilibria presented later, prices $p_1, \dots, p_T$ often reveal $t_0$ exactly. However, in some instances, this will hold only approximately, and a few values of $t_0$ will be consistent with prices. Nevertheless, to avoid burdening the reader with inessential complications, I will assume that $t_0$ is exactly revealed. Generalizing the fundamental-value formula to take the latter cases into account adds complications in exchange for little or no insight.

In equilibrium, for all $i$, $a_{i,t}$ is optimal given agent $i$'s shock realization $\delta_{i,t}$ and the belief $\mu_{i,t}(t_0)$, and $\mu_{i,t}(t_0)$ is consistent with the equilibrium strategy profile. To be consistent with a strategy profile, a belief $\mu_{i,t}(t_0)$ must assign positive probability only to values of $t_0$ that are not ruled out by strategies, given $p^{t-1}$ and if $t \geq \nu(i)$, $\nu(i)$. The set of values of $t_0$ that are not ruled out is the support of $t_0$, denoted by $\text{supp}_{i,t}(t_0)$. Since beliefs are the same for all agents in a type, we can define $\text{supp}_{n,t}(t_0) \equiv \text{supp}_{i,t}(t_0)$ for all $i$ with $\nu(i) = n$. To see how $\text{supp}_{n,t}(t_0)$ evolves in equilibrium, recall that the signal $n$ implies that $\text{supp}_{n,t}(t_0) \subseteq \{\max\{1, n-(N-1)\}, \ldots, n\}$. Moreover, prices $p^{t-1}$ and strategies rule out values of $t_0$ as follows. If $t_0$ takes on the value $\tau_0$, given the price history $p^{t-1}$, there are—discount-factor contingent—implied values of $a_{i,\tau}$ for all $i$ and all $\tau < t$. These implied actions and the price $p_\tau$ can be substituted into (17) to compute the implied $\varepsilon_\tau$. The value $\tau_0$ is excluded from $\text{supp}_{n,t}(t_0)$ if it implies $|\varepsilon_\tau| > \bar{\varepsilon}$ for some $\tau$. Once an agent has discarded the values of $t_0$ that are ruled out by this process, the probabilities that $\mu_{n,t}(t_0)$ assigns to the values in $\text{supp}_{n,t}(t_0)$ are obtained using Bayes' rule.

With equilibrium beliefs embeeded in the expectations operator $E_{i,t}$, the equilibrium strategy $a_{i,t}$ solves the following problem for all agents and at all times

$$V(b_{i,t}, h_{i,t}) = \max_{m_{i,t}, s_{i,t}, \chi_{i,t}} \; E_{i,t}\left[c_{i,t}\right] + \delta_{i,t} E_{i,t}\left[V(b_{i,t+1}, h_{i,t+1})\right], \tag{18}$$

subject to (5)-(8), (12), (13), and ***Restriction I***. As previously stated, preference shocks and risk neutrality greatly simplify this program's solution. Agents hit by the shock set $a_{i,t} = (0, h_{i,t}, 1)$, i.e., sell and consume everything. Agents with $\delta_{i,t} = 1/R$ set $\chi_{i,t} = 0$. Depending on whether $E_{i,t}\left[p_{t+1}/p_t\right]$ is above, below, or equal to $R$, they are, respectively, fully invested in the risky asset, fully invested in the risk-free asset, or indifferent between any mix of the two assets.

*[The following Sections 4 and 5 are incomplete/under revision. The Lemmas and Propositions that follow will continue to hold, and convey essentially the same message, when revised. However, to incorporate features of the environment, many details need to be updated.]*

# 4 Equilibria with Bubbles: Basic Analysis

I begin the analysis in subsection 4.1 by considering the case where $\bar{\varepsilon} < (1-\bar{\theta})/(2N-1)$. This is such a low level of noise that the price is certain to reveal sales as soon as one type $(z_t = 1)$ exits the market. In this effectively noiseless environment, I examine the possibility of bubbly equilibria with simple trigger strategies akin to those analyzed by AB, where agents attempt to ride the bubble for $\tau^* \geq 0$ periods. Specifically, equilibrium strategies dictate that—unless the preference shock forces them to sell or the bubble bursts first—agent *i* shall be fully invested in the risky asset until selling out at $t = \nu(i) + \tau^*$. Although noise cannot hide any sales, the discreteness of the model makes it possible for a positive mass $1/N$ of agents to sell simultaneously before the crash. The price will reveal these sales and precipitate a crash, but by then the first sellers will already be out of the market. Since there is a nonzero probability of being among these first sellers, as long as the bubble grows fast enough, it is optimal to try to ride it.[14] Not surprisingly, there is a positive relationship between the speed at which the bubble grows, captured by $G/R$, and bubble duration $\tau^*$, and there is also a minimum threshold level $\Gamma$ below which there can be no bubbles without noise. In Proposition 1, I derive a parameter restriction such that, if $\bar{\varepsilon} < (1-\bar{\theta})/(2N-1)$, the only equilibrium is one where agents sell immediately upon observing the signal. Specifically, if $e^{\lambda} < G/R < \Gamma$, where the threshold $\Gamma$ depends on $\lambda$ and $N$, only $\tau^* = 0$ is an equilibrium.

In subsection 4.2, I increase the level of $\bar{\varepsilon}$ so that noise can hide sales of one type, but not two. For levels of noise in this range, prices can be categorized as *high, medium*, or *low.* High prices reveal that no types have left the market, medium prices are consistent with either no sales or with sales by one type, and low prices reveal for sure that some types have left the market. I focus on strategies that are Markovian, in the sense that they condition actions—besides the preference shock—exclusively on the last price observed, not the entire price history. Even with this minimal amount of noise, there exist parameters such that bubbles of arbitrary length can arise, even for levels of $G/R$ below the threshold level $\Gamma$.

[14] In continuous time, sales by an arbitrarily small mass of agents would be revealed by the price. This would drive the probability of being one of the first sellers to zero, and thus riding the bubble would not be optimal. To avoid this, AB assume that there is an interval of time during which shares are gradually sold without affecting the price, which grows at the same rate as before sales began, and reacts only when total sales reach a threshold $\kappa \in (0,1)$.

## 4.1 The Case where Noise Can Hide No Sales

Let $\bar{\varepsilon} < (1-\bar{\theta})/(2N-1)$, so that sales are always detected, as $((\bar{\theta}+\bar{\varepsilon})^{-1}-1))G^t$, the lowest possible price before any type has left the market exceeds $([1/N+(\bar{\theta}-\bar{\varepsilon})(1-1/N)]^{-1}-1)G^t$, the highest possible price if one type has sold. Consider the following strategies:

***<u>Strategy Profile 1</u>*** *— For any $i \in [0,1]$, agent i follows $\{a_{i,t}\}_{t\in\mathbb{Z}} = \{m_{i,t}, s_{i,t}, \chi_{i,t}\}_{t\in\mathbb{Z}}$ given by:*

- *If $\delta_{i,t} = 0$,* $\quad a_{i,t} = (0, h_{i,t}, 1)$ *for any t.*
- *If $\delta_{i,t} = 1/R$,* $\quad \chi_{i,t} = 0$ *for all t, and the choice of $(m_{i,t}, s_{i,t})$, is as follows:*
    - *If $t \le T$,*

$$(m_{i,t}, s_{i,t}) = \begin{cases} (b_{i,t}+e_t, 0) & \text{if } \ t < \nu(i)+\tau^* \\ (0, h_{i,t}) & \text{if } \ t \ge \nu(i)+\tau^*, \end{cases} \tag{19}$$

*with $\tau^* \ge 0$.*

- *If $t \ge T+1$,* $(m_{i,t}, s_{i,t}) = ((R/G)^{t-(t_0-1)} e_t, 0)$.

In words, the strategy dictates the following: When hit by the shock, agent *i* shall sell all her assets and consume the proceeds. When not hit by the shock, she shall not consume and allocate her savings as follows. Before the crash, invest as much as possible into the risky asset if less than $\tau^*$ periods have passed since observing the signal, and sell all shares of the risky asset at time $\nu(i)+\tau^*$. After the crash, she shall invest a fraction $(R/G)^{t-(t_0-1)}$ of her endowment into the risky asset and the rest into the risk-free asset.

If all agents follow these strategies, in equilibrium, type-$t_0$ agents sell at $t_0+\tau^*$, and are the only agents who succeed in riding the bubble and selling before the crash, as $p_{t_0+\tau^*}$ reveals their sales, and the crash happens at time $T+1 = t_0+\tau^*+1$.

In Proposition 1, I show that, if $e^{\lambda} < G/R < \Gamma$ (where $\Gamma = e^{\lambda}(1+\sqrt{1+4e^{-\lambda}})/2$), agents must sell as soon as they observe the signal. That is, agents are willing to follow (19) if and only if $\tau^* = 0$.[15] Inequality $e^{\lambda} < G/R$ ensures that agents do not sell before observing the signal, while inequality $G/R < \Gamma$ ensures that they sell as soon as they observe it. If $G/R < \Gamma$, waiting

[15] Technically, $\tau^* = \infty$ is also an equilibrium, but it is not of interest, since the assumption of perpetually fast endowment growth shall not be taken literally.

for $\tau^* \geq 1$ periods after observing the signal cannot be optimal, because the bubble is not growing fast enough to compensate for potential losses in the event of a crash.

**<u>Proposition 1</u>** *If* $e^{\lambda} < G/R < \Gamma,$ *where* $\Gamma = e^{\lambda}(1+\sqrt{1+4e^{-\lambda}})/2,$ ***Strategy Profile 1*** *and its implied beliefs constitute an equilibrium if and only if* $\tau^* = 0.$

**<u>Proof</u>** Because agents who are hit by the shock do not value the future, they always find it optimal to sell and consume everything they own. Keeping this in mind, henceforth, the proof will eximne the optimality of decisions for agents who are not hit by the shock.

Pre-boom and post-crash, for any parameters and $\tau^* \geq 0,$ agents are willing to follow (19). While $t \leq 0,$ they are indifferent between any holdings of the risky asset, and thus, being fully invested in the risky asset by letting $(m_{i,t}, s_{i,t}) = (b_{i,t} + e_t, 0)$ is weakly optimal. Similarly, in post-crash periods $t \geq T+1,$ the risky and riskless assets are again perfect substitutes, and thus, a $(m_{i,t}, s_{i,t}) = ((R/G)^{t-(t_0-1)} e_t, 0)$ is also weakly optimal. For boom periods $t \in \{1, \ldots, T\},$ the proof has two parts. First, I show that, if $e^{\lambda} < G/R,$ there is an equilibrium with strategies given by (19) and $\tau^* = 0.$ Second, I show that, if $G/R < \Gamma,$ there are no equilibria with strategies given by (19) and $\tau^* \geq 1.$

For the first part, note that, in equilibrium with $\tau^* = 0,$ type-$n$ agents *(i)* find it optimal to sell in period $n,$ and *(ii)* find it optimal not to sell before period $n.$ To see why *(i)* holds, note that, at time $n$ $(= n + \tau^*),$ a type-$n$ agent can infer that $t_0 = n$ from the fact that the bubble has not burst. She also knows that other type-$n$ agents are selling, and that $p_n$ will reveal these sales, causing a crash at $n+1.$ Clearly, selling is optimal since the expected time-$n$ price $G^n$ exceeds the discounted post-crash price $G^{n-1}$ that she will get if she waits. Next, to see why $e^{\lambda} < G/R$ implies *(ii)*, consider a type-$n$ agent at $t < n.$ If $t \leq 0,$ she has no reason to sell, since $t_0$ cannot be $t,$ and a crash at $t+1$ is impossible. If $t \geq 1,$ $t_0$ could be $t,$ and thus the bubble could burst at $t+1.$ With $\text{supp}_{n,t}(t_0)$ given by $\{\tau \mid \tau \geq t\},$ the probability of a crash at $t+1$ is $\mu_{n,t}(t) = 1 - e^{-\lambda}.$ The agent can sell at a price $G^t,$ or she can wait, in which case with probability $e^{-\lambda}$ she will be able to sell at $t+1$ for a higher (discounted) price $G^{t+1}/R$ and with probability

$1-e^{-\lambda}$ she will obtain the post-crash price. Even if the post-crash price is zero, if $1<e^{-\lambda}G/R,$ waiting is optimal. Hence, $e^{\lambda}<G/R$ suffices to rule out preemptive sales while $t<n.$

For the second part (i.e., showing that there are no equilibria with $\tau^*\geq 1$ if $G/R<\Gamma$), suppose, by means of contradiction, that there is such an equilibrium although $G/R<\Gamma.$ In any equilibrium with $\tau^*\geq 1,$ type-$n$ agents must be willing to wait at all times $t<n+\tau^*,$ including at $t=n+\tau^*-1.$ If $t=n+\tau^*-1$ and the bubble has not burst, type-$n$ agents know that their type was either first or second to observe the signal, i.e., $\text{supp}_{n,t}(t_0)=\{n-1,n\}.$ By Bayes' rule, $\mu_{n,t}(n-1)=1/(1+e^{-\lambda})$ and $\mu_{n,t}(n)=e^{-\lambda}/(1+e^{-\lambda}).$ In this situation, a type-$n$ agent's sell-or-wait trade-off is as follows. Selling preemptively at $t$ yields $G^{n+\tau^*-1},$ while waiting yields the (discounted) post-crash price $G^{n-2}R^{\tau^*+1}$ if $t_0=n-1,$ and $G^{n+\tau^*}/R$ if $t_0=n.$ In sum, deviating from (19) by selling preemptively at $t$ is optimal if

$$1>\frac{1}{1+e^{-\lambda}}\left(\frac{G}{R}\right)^{-(\tau^*+1)}+\frac{e^{-\lambda}}{1+e^{-\lambda}}\frac{G}{R}. \tag{20}$$

Since the right hand side is decreasing in $\tau^*,$ if (20) holds for $\tau^*=1,$ it also does for $\tau^*>1$. In Appendix A, I show that $1+e^{-\lambda}>(G/R)^{-2}+e^{-\lambda}G/R$ holds if $1<G/R<\Gamma,$ where $\Gamma=(1+\sqrt{1+4e^{-\lambda}})e^{\lambda}/2.$ Thus, there are no equilibria with $\tau^*\geq 1$ and $G/R<\Gamma.$ **Q.E.D.**

## 4.2 The Case where Noise Can Hide One Sale

To fix ideas and lay groundwork, let us provide details on the case where noise can hide sales by one type, but not two. Given (14) and (16), if $\bar{\varepsilon}>(1-\bar{\theta})/(2N-1),$ noise may hide sales by one type, but it cannot hide simultaneous sales by two types if $\bar{\varepsilon}<(1-\bar{\theta})/(N-1).$ The price must also reveal sales if the second type sells after the first. This will be the case if

$$\frac{1-\bar{\theta}-\bar{\varepsilon}}{\bar{\theta}+\bar{\varepsilon}}>\left(\frac{N-1}{N}\right)\frac{1-\bar{\theta}+\bar{\varepsilon}}{\bar{\theta}-\bar{\varepsilon}+1/(N-2)},$$

since this inequality implies that the lowest possible price when nobody has sold is greater than the highest possible price given by (17) with $z_t=z_{t+1}=1.$ Solving for $\bar{\varepsilon},$ we find that if

$$\bar{\varepsilon}_R\equiv\left(N+\frac{1}{N-2}\right)-\sqrt{\left(N+\frac{1}{N-2}\right)^2-\left(1-\bar{\theta}\right)\left(\frac{N}{N-2}+\bar{\theta}\right)}.$$

that the price will fall below $G^t(1-\overline{\theta}-\overline{\varepsilon})/(\overline{\theta}+\overline{\varepsilon})$ revealing the sales, and a probability that the price will remain above $G^t(1-\overline{\theta}-\overline{\varepsilon})/(\overline{\theta}+\overline{\varepsilon})$

Maintaining the restriction that $e^{\lambda} < G/R < \Gamma,$ which precludes bubbles without noise, I now increase $\overline{\varepsilon}$ and derive conditions under which bubbles arise. Despite restrictions **I-IV**, when $\overline{\varepsilon} > 1/N,$ multiple equilibria appear. Nevertheless, since all equilibria with long bubbles share certain features, the analysis points to a set of conditions that are necessary for bubbles. In bubbly equilibria, the higher prices are, the longer are agents willing to postpone their sales after observing the signal. Since different prices elicit different behavior, price fluctuations reveal information about the value of $t_0,$ i.e., there is gradual *informational leakage*, as in Kai and Conlon (2008). For intstance, a recovery after a price slowdown reveals that $t_0$ exceeds a certain threshold, since the slowdown would have triggered more sales if $t_0$ was lower. By contrast, a string of consecutive high prices is consistent with $t_0$ being quite low, in which case several types would be awaiting the next slowdown, ready to sell. Confidence in the bubble is thus strongest after a recovery and weakest when, after many high prices, there is a slowdown.[16] To support bubbles in equilibrium, agents must be willing to postpone sales for some time after observing their signal. During this time, it must be optimal for agents to wait even if they see a price slowdown. Roughly, sufficient conditions to rule out such preemptive sales can be stated as follows. First, there must be enough noise to imply a sizable probability that, when the first agents sell, the price does not reveal the sales. Second, agents who sell preemptively during a slowdown must forgo a large profit if the bubble continues to grow after they have sold. This forgone profit is large if price slowdowns are not too frequent and bubble growth is fast enough.

To make these ideas precise, consider the case where $1/N < 2\overline{\varepsilon} < 2/N,$ so that noise can hide sales by one type, but not two.[17] A price $p_t$ can be in one of three categories, which I will

[16] This is similar to the version of AB with exogenous price drops or sunspots, where recoveries after price drops also reveal information, and multiple equilibria are inevitable since price drops may trigger sales or be ignored. However, unlike in AB, here price drops can be due to noise or sales, and agents take this into account in their inference. Furthermore, because time is discrete in this model, if agents exit the market and reenter when they see that prices recover, they miss out on a nontrivial amount of profits.

[17] Since, as previously mentioned, within-period timing assumptions are plausible only if periods are relatively brief, this is a restrictive assumption. However, analyzing this case is useful for expositional purposes since it is more tractable than the case where noise may hide sales by multiple types. (The latter is available upon request.)

refer to as *high*, *medium* and *low*, respectively. High prices exceed $G^t + \alpha(\bar{\varepsilon} - 1/N),$ and thus reveal that nobody has sold. Low prices are under $G^t - \alpha\bar{\varepsilon},$ and thus reveal that sales have begun. Medium prices are between these two thresholds, and are therefore consistent with nobody having sold and with one type having sold. Before sales begin $(H_t = 1),$ $p_t$ is high with probability

$$\pi = \frac{1/N}{2\bar{\varepsilon}} \tag{21}$$

and medium with probability $1-\pi.$ With one type out of the market $(H_t = 1 - 1/N),$ $p_t$ is low with probability $\pi$ and medium with probability $1-\pi.$ If at least two types sell, $p_t$ must be low. Also note that, since $1/N < 2\bar{\varepsilon} < 2/N,$ $\pi \in (½,1).$

Following restrictions **I-IV**, type-*n* agents condition their sell-or-wait choice at time $t$ on time since observing the signal, i.e., on $t-n,$ and on whether the last price $p_{t-1}$ was high $(c(p_{t-1}) = 0)$ or medium $(c(p_{t-1}) = 1)$. Specifically, agents' plans are described by:

***Strategy Profile 2*** *— For any* $n \geq 1,$ *the strategy of a type-n agent is the following:*

- *For all* $t \leq 1,$ *hold* $h_{n,t} = 1.$
- *For all* $t \in \{2, \ldots, T\},$ *let*

$$h_{n,t} = \begin{cases} 1 & \text{if } \ t < \min\{t^*(n), t^{**}(n)\} \\ 0 & \text{if } \ t \geq \min\{t^*(n), t^{**}(n)\}, \end{cases} \tag{22}$$

  *where* $t^*(n) = \min\{t \mid t \geq n + \tau^* \ \wedge \ c(p_{t-1}) = 1\},$ $t^{**}(n) = \min\{t \mid t \geq n + \tau^{**} \ \wedge \ c(p_{t-1}) = 0\},$ *and* $\tau^{**} \geq \tau^* \geq 0.$
- *For all* $t \geq T+1,$ *maintain* $h_{n,t} = h_{n,t-1}.$

In words, type-*n* agents hold the maximum long position before observing the signal (i.e., while $t < n$). After observing the signal, they wait for $\tau^*$ periods until $t = n + \tau^*,$ then sell if $p_{n+\tau^*-1}$ is medium and wait if it is high.[18] They continue applying this sell-if-medium/wait-if-high rule for another $d$ periods, where $d \equiv \tau^{**} - \tau^*.$ In the event that prices remain high for all

[18] In the special case where $n + \tau^* = 1,$ the sell-if-medium/wait-if-high rule does not apply, as $c(p_0)$ is not defined. In this case, type-1 agents do not sell at $t = 1;$ they begin to follow the sell-if-medium/wait-if-high at time 2 instead.

$t \in \{n+\tau^*-1,\ldots,n+\tau^{**}-1\}$, they sell at $n+\tau^{**}$, even though $p_{n+\tau^{**}-1}$ is high. Finally, agents do not reenter the market after selling, and nobody does anything after time $T$.

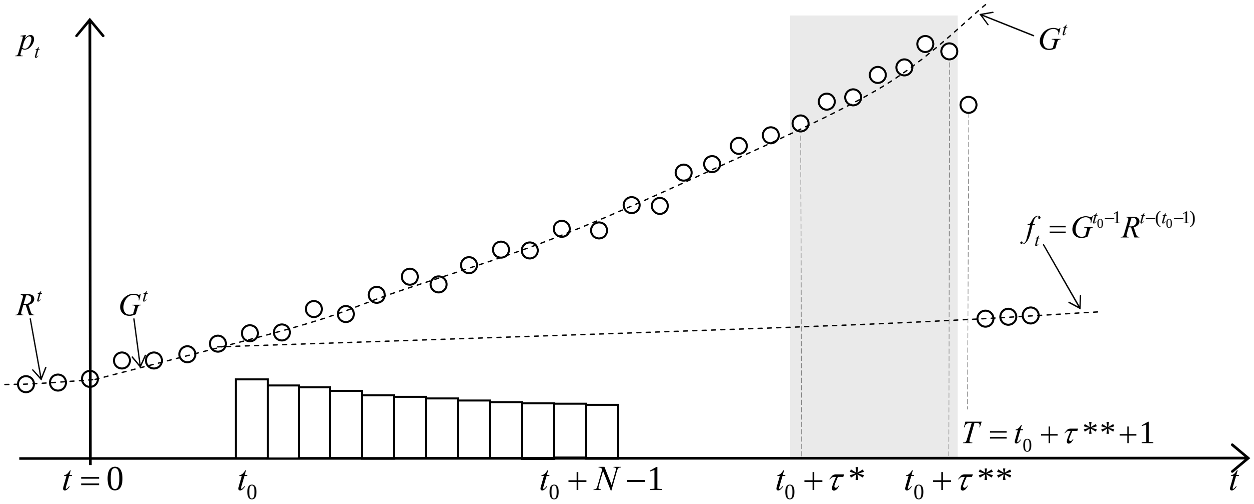


**Figure 4** — As soon as $t_0$ is realized, $p_t$ and $f_t$ begin to diverge. Signals are observed from $t_0$ to $t_0+N-1$. (Bars above these periods, which decrease in height, denote conditional probabilities for the signal $n = t_0 + N - 1$.) In this example, since $\tau^* > N-1$, sales cannot begin until after all signals are observed. Also, $d = \tau^{**} - \tau^* = 6$. For the depicted realizations of $\varepsilon_t$, the bubble bursts as late as possible. Since $p_t$ is high $\forall t \in \{t_0+\tau^*-1,\ldots,t_0+\tau^{**}-1\}$, sales begin at time $t_0+\tau^{**}$. Since $p_{t_0+\tau^{**}}$ is medium, types $t_0+1,\ldots,t_0+7$ sell at $T = t_0+\tau^{**}+1$, causing a crash. If $p_\tau$ had been medium for some $\tau \in \{t_0+\tau^*-1,\ldots,t_0+\tau^{**}-2\}$ sales would have started at time $\tau+1$, before $t_0+\tau^{**}$.

Depending on $\varepsilon_t$, sales can begin as soon as period $t_0+\tau^*$, and as late as $t_0+\tau^{**}$. The number of types that manage to sell before the crash ranges from 1 to $d+2$ and also depends on the realizations of $\varepsilon_t$ in periods leading up to the crash. Specifically, if $p_{t_0+\tau^*-1}$ is medium, type $t_0$ sells at $t_0+\tau^*$. If $p_{t_0+\tau^*}$ is low, no one else sells, whereas if it is medium, type $t_0+1$ sells at $t_0+\tau^*+1$. If $p_{t_0+\tau^*-1}$ is high, the next $s$ prices (with $s \in \{0,1,\ldots,d-1\}$) are high and $p_{t_0+\tau^*+s}$ is medium, $s+2$ types sell at $t_0+\tau^*+s$. And if for all $t \in \{n+\tau^*-1,\ldots,n+\tau^{**}-1\}$, prices remain high, type $t_0$ sells at $t_0+\tau^{**}$. Next, if $p_{t_0+\tau^{**}}$ is low, no more types sell, but if $p_{t_0+\tau^{**}}$ is medium—as shown above in Figure 4—another $d+1$ types also sell before the crash. As previously discussed, behavioral agents can only buy a mass $\kappa < 1$ of shares. Thus, $d+2$ types can sell before the crash only if $d+2<\kappa N.$ I assume that this holds.

To investigate the model's ability to generate long bubbles, note that equilibrium bubble duration $T - t_0$ is at least $\tau^*+1$ periods. Thus, the task at hand is to find conditions under which equilibria with large $\tau^*$ (and hence large $\tau^{**}$) can be supported. In any equilibrium, agents must be willing to sell if and only if (22) stipulates it. As discussed before, pre-boom ($t \le 0$) and post-crash $(t \ge T+1)$, all types find it (weakly) optimal to follow equilibrium strategies. And at $t=1,$ they find it optimal not to sell, since nobody is selling and they can reap gains postponing their sales by at least one period. The analysis of agents' choices while $t=2,\dots,T$ is more complex, and I therefore divide it into four parts: In Lemma 1, I state conditions under which type-*n* agents choose to sell if $t=n+\tau^{**}$ and $p_{t-1}$ is high. In Lemma 2, conditions such that they choose to wait if $t<n+\tau^{**}$ and $p_{t-1}$ is high, and in Lemmas 3 and 4, respectively, conditions such that they are willing to sell if $t \ge n+\tau^*$ and $p_{t-1}$ is medium, and wait if $t<n+\tau^*$ and $p_{t-1}$ is medium. In Proposition 2, I combine the Lemmas into a set of conditions that are necessary to support equilibria with large $\tau^*$ and $\tau^{**}.$ Finally, in Proposition 3, I show that the conditions in Lemma 2 are compatible with each other and with the no-bubbles-without-noise restriction $e^{\lambda} < G/R < \Gamma.$

Before plunging into the Lemmas, I will provide a brief preview of upcoming results. Making agents sell is easy. If a type-*n* agent knows that other type-*n* agents are selling, there is at least a probability $\pi$ that the bubble will burst next period. Given this, Lemmas 1 and 3 require very little to make agents sell when they are supposed to. It is more difficult to make agents wait when they are supposed to wait, and therefore, Lemmas 2 and 4 need to impose some restrictions to rule out preemptive sales. In Lemma 2, it is only possible to support large $\tau^{**}$ if $G/R$ is above (or not far below) $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi).$ In Lemma 4, it is only possible to support large $\tau^*$ if $e^{\lambda}/\pi < G/R.$ Under these conditions, agents are willing to wait because, if they sell early and the bubble continues to grow, they forgo large profits. To see exactly how these conditions affect agents' choices, let us proceed to the Lemmas.

**Lemma 1** — *Let* $n \geq 1$ *be an arbitrary type. If* $G/R < \Gamma$ , $\pi \geq 1/(1+e^{-\lambda})$, $d+2 < \kappa N$, $\tau^{**} \geq 1$, $t = n + \tau^{**}$ *and* $p_{t-1}$ *is high, type-n agents find it optimal to sell at time* $t$.

**Proof** — Consider a type-*n* agent at $t = n + \tau^{**}$, with $\tau^{**} \geq 1$ and high $p_{t-1}$. At this point, she knows that her type is first, i.e., that $t_0 = n$. (If $n = 1$, she has known that $t_0 = n$ since period 1; if $n > 1$, she learnt that $t_0 = n$ from the fact that $p_{n+\tau^{**}-1}$ was high.) Since other type-$t_0$ agents are selling at $t$, $p_t$ will be low with probability $\pi$ and medium with probability $1-\pi$. An individual type-$t_0$ agent can thus sell along with the other type-$t_0$ agents at an expected price $G^{t_0+\tau^{**}}$, or wait. If she waits, with probability $\pi$ she will earn the discounted post-crash price $(G/R)^{t_0-1} R^{t_0+\tau^{**}}$ and with probability $1-\pi$ she will sell at $t+1$ with $d+1$ other types at an expected price that—since $d+2 < \kappa N$ —equals $G^{t_0+\tau^{**}+1}$. Thus, selling is optimal if

$$1 \geq \pi \left( \frac{G}{R} \right)^{-(\tau^{**}+1)} + (1-\pi) \frac{G}{R}. \tag{23}$$

If $\pi = 1/(1+e^{-\lambda})$, (23) is the same as (20). Thus, since $\pi \geq 1/(1+e^{-\lambda})$, $G/R < \Gamma$ and $\tau^{**} \geq 1$, type-*n* agents are willing to sell. **Q.E.D.**

In Lemma 1, I have ignored the case where $\tau^{**} = 0$. Analyzing this case is not difficult, but it is tedious and, since our focus in on long bubbles, uninteresting. Similarly, assuming that $\pi \geq 1/(1+e^{-\lambda})$ simplifies the proof without actually imposing a binding restriction on parameters. This is because, as we will see in Proposition 3, to support long bubbles $\pi$ must be above (or just a little bit below) a threshold $(1+e^{-\lambda})/(1+e^{-\lambda}+e^{-2\lambda})$. Since this threshold exceeds $1/(1+e^{-\lambda})$ for any $\lambda$, the parameter values of interest always satisfy $\pi \geq 1/(1+e^{-\lambda})$.

Lemma 1 establishes that, under general conditions, type-*n* agents will sell at $n + \tau^{**}$ after a high $p_{n+\tau^{**}-1}$. At this point, they know that they were the first to observe the signal and that they have successfully ridden the bubble. But how did they arrive here? In earlier periods $n + \tau^{**} - j$ (with $j \geq 1$), they did not know that they were first. Following (22) and waiting was risky, since $t_0$ could have been $n - j$, in which case type $n - j$ would have sold at $n + \tau^{**} - j$, causing a crash with probability $\pi$. Under what conditions was it optimal for them to take this risk? While I answer this fully in Lemma 2, the key condition ruling out preemptive sales can be understood by examining Figure 4 and focusing on period $t_0+\tau^{**}$, which is preceded by $d+1$ high prices $p_{t_0+\tau^*-1},\ldots,p_{t_0+\tau^{**}-1}$. As Figure 4 shows, in this situation, type-$t_0$ agents sell and others wait. To see why others wait, consider the sell-or-wait trade-off of type $t_0+1$, i.e., the second type to observe the signal.[19] At $t=t_0+\tau^{**}$, type-$t_0+1$ agents understand that, with probability $1/(1+e^{-\lambda})$, they were second to observe the signal and the first type will sell at $t$, causing a crash with probability $\pi$. But they also assign a probability $e^{-\lambda}/(1+e^{-\lambda})$ to the possibility that they were first, in which case nobody will sell at $t$ and a crash at $t+1$ is impossible. In sum, selling at $t$ yields $G^t$, and waiting yields the post-cash price $(G/R)^{t_0-1}R^t$ with probability $\pi/(1+e^{-\lambda})$ and at $G^{t+1}$ with probability $1-\pi/(1+e^{-\lambda})$. Waiting is optimal if

$$1\le\frac{1}{1+e^{-\lambda}}\left(\pi\left(\frac{G}{R}\right)^{-(\tau^{**}+1)}+(1-\pi)\frac{G}{R}\right)+\frac{e^{-\lambda}}{1+e^{-\lambda}}\frac{G}{R}. \tag{24}$$

If $G/R\ge(1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$, (24) holds for any $\tau^{**}$, no matter how large. In other words, for $G/R$ above this threshold, type-$t_0+1$ agents are willing to wait even if the post-crash price is zero. If $G/R$ is below $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$, waiting is optimal only if the fraction of the price lost in the crash is not too large. Specifically, waiting is optimal only if

$$\tau^{**}\le\frac{\ln\left(\dfrac{\pi}{(1+e^{-\lambda})-(1+e^{-\lambda}-\pi)G/R}\right)}{\ln\left(G/R\right)}-1. \tag{25}$$

I have derived (24) for the specific situation of type-$t_0+1$ agents at $t_0+\tau^{**}$ after $d+1$ high prices. However, in the proof of Lemma 2, I show that, of all possible situations with $t<n+\tau^{**}$ and high $p_{t-1}$, this is precisely the one where type-$n$ agents are most tempted to sell.

**<u>Lemma 2</u>** — *Suppose that* $e^{\lambda}<G/R$, $d+2<\kappa N$, $t<n+\tau^{**}$ *and let* $p_{t-1}$ *be high. If* $G/R\ge(1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$, *type-n agents find it optimal not to sell at time* $t$, *for any* $\tau^{**}\ge 0$. *If* $G/R<(1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$, *type-n agents choose not to sell at time* $t$, *only if* (25) *holds.*

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[19] Note that the trade-off that type-$t_0$ agents face at $t_0+\tau^{**}-1$, with high prices for the last $d+1$ periods, is identical to the one faced by type-$t_0+1$ agents at $t_0+\tau^{**}$, also with high prices for the last $d+1$ periods. In both cases, the support of $t_0$ has two values; agents know that they were either be first or second to observe the signal.

**Proof** — *See Appendix B.*

While I refer the reader to Appendix B for analysis of all possible cases with $t < n + \tau^{**}$ and high $p_{t-1}$, the reason why the situation captured by (24) is the one where preemptive selling (after a high price) is most tempting can be sketched as follows. Consider type-$t_0 + 2$ agents at time $t_0 + \tau^{**}$ after $d+1$ high prices. Given their information, they believe they could have been first, second, or third to observe the signal. Thus, for them, the crash probability is $\pi / (1 + e^{-\lambda} + e^{-2\lambda})$ ($\pi$ times the probability of being third), clearly below $\pi / (1 + e^{-\lambda})$, the crash probability in (24). By the same logic, for types $t_0 + 2$ and higher, the crash probability is even lower. Moreover, in many cases with $t < n + \tau^{**}$ and high $p_{t-1}$, type-*n* agents have no incentive to sell, since the crash probability is nil. For instance, if $p_{t-s}$ is medium for some $s \in \{2, \ldots, d+1\}$, all types know with certainty that nobody will sell at $t$, and hence that a crash at $t+1$ is impossible. This is due to the fact that, if $t_0$ was $t - \tau^{**}$, type-$t_0$ agents would have sold at $t - s + 1$ after the medium $p_{t-s}$. But then, $p_{t-1}$ could not possibly be high.

From (25), we can analyze comparative statics for maximum bubble duration $\tau^{**}$. Not surprisingly, $\tau^{**}$ falls as $\lambda$ increases, since the greater $\lambda$, the greater the likelihood of lower values of $t_0$ relative to higher values. The effect of $\pi$ on $\tau^{**}$ is also negative, since the greater $\pi$, the more likely it is that, if one type sells at $t$, the price reveals the sale.[20] The effect of $G/R$ is not as straightforward, because increases in $G/R$ increase profits if the bubble does not burst, but also losses if it does burst. In general, the direction of the effect depends on parameter values. However, the parameter values of interest are those that allow long bubbles to arise, i.e., values of $G/R$ slightly below $(1+e^{-\lambda})/(1+e^{-\lambda} - \pi)$. In this range, $\tau^{**}$ is increasing in $G/R$.

In sum, by Lemma 2, equilibria with large $\tau^{**}$ exist only if $G/R$ is above, or not far below, $(1+e^{-\lambda})/(1+e^{-\lambda} - \pi)$, and if $d + 2 < \kappa N$. The latter condition, which is needed to ensure that behavioral agents can buy the $(d+2)/N$ shares sold by rational agents, can only hold if

[20] These comparative statics resemble those in AB (abstracting from AB's exogenous cap on bubble duration). For some parameter values, there is no endogenous upper bound on bubble duration. For other values, there is a finite endogenous bubble duration, which is increasing in the rate of growth of the bubble and decreasing in $\lambda$ ($\lambda$ has the same meaning in AB and here). The fact that, in this model, $\tau^{**}$ is decreasing in $\pi$ corresponds loosely to the fact that, in AB, bubble duration is increasing in behavioral absorption capacity. Here, the higher $\bar{\varepsilon}$, (and hence, the lower $\pi$) the more agents can sell unnoticed.

$\tau^*$ is not too far below $\tau^{**}$. To investigate what values of $\tau^*$ can arise in equilibrium, in Lemmas 3 and 4, I study sell-or-wait tradeoffs after medium prices.

Lemma 3 is fairly straightforward. In Lemma 1, type-*n* agents sold at $n+\tau^{**}$ with high $p_{n+\tau^{**}-1}$ knowing that they were first to observe the signal, and the only ones selling. Thus, the crash probability was $\pi$. By contrast, in Lemma 3, type-*n* agents sell at $n+\tau^*$ after medium $p_{n+\tau^*-1}$, in general not knowing whether they were first to observe the signal or not. Besides their own type, others could also be selling, and this raises the crash probability above $\pi$. Agents are thus more inclined to sell after medium prices than after high prices, which implies that the conditions that sufficed to induce sales in Lemma 1 also suffice in Lemma 3.

**Lemma 3** — *If* $\pi \geq 1/(1+e^{-\lambda})$, $d+2<\kappa N$, $G/R<\Gamma$, $\tau^* \geq 0$ *and* $n+\tau^* \geq 2$, *type-n agents are willing to sell at time* $t \geq n+\tau^*$ *after a medium price* $p_{t-1}$.

**Proof** — First consider the case with $n \geq 2$ and $t = n+\tau^*$. In Lemma 1, type-*n* agents sell knowing that $t_0 = n$. Here, since $p_{t-1}$ is medium, they sell without knowing whether $t_0 = n$ or $t_0 < n$.[21] If $t_0 = n$, the bubble will burst at $t+1$ with probability $\pi$. And if $t_0 < n$, two or more types will sell at $t$, causing a crash at $t+1$. The probability of a crash at $t+1$ in this case is thus above $\pi$. This makes incentives to sell stronger at $n+\tau^*$ with medium $p_{n+\tau^*-1}$ than at $n+\tau^{**}$ with high $p_{n+\tau^{**}-1}$. Since $\pi \geq (1+e^{-\lambda})^{-1}$ and $G/R<\Gamma$ suffice to make agents sell in the latter case, they also suffice in the former.

Continuing with $n \geq 2$, if $t > n+\tau^*$ and $p_{t-1}$ is medium, type-*n* agents sell at $t$, since they know that at least two types ($n$ and $n+1$) will sell at $t$, which ensures a crash at $t+1$.[22]

Finally, let $n=1$. If $\tau^*>0$ and $p_{(1+\tau^*)-1}$ is medium, type-1 agents sell at $t=1+\tau^*$, knowing that they are the only type selling. By Lemma 1, selling is optimal because $\pi \geq 1/(1+e^{-\lambda})$, $G/R<\Gamma$, and $\tau^* \geq 1$. If $\tau^*>0$ and $p_{(1+\tau^*)-1}$ is high (or if $\tau^*=0$), type-1 agents

[21] If $n \geq 2$ and $p_{t-1}$ is medium, type-*n* agents cannot rule out the possibility that $t_0 = n-1$. This is true regardless of whether $p_{t-2}$ is high or medium. It also holds if $c(p_{t-2})$ is not defined, i.e., if $t=2$ with $n=2$ and $\tau^*=0$. Thus, $\text{supp}_{n,t}(t_0)$ has at least two elements $\{n-1, n\}$. In may have even more, since, if $p_{t-1}$ is the first medium price after $k \geq 2$ consecutive high prices, $\text{supp}_{n,t}(t_0)$ is given by $\{n-k, \ldots, n\}$.

[22] Type-*n* agents find themselves in this situation if $p_{n+\tau^*-1}$ happens to be high.

sell at $t > 1+\tau^*,$ as soon as $p_{t-1}$ is medium. In these cases, selling is optimal because two or more types are selling and a crash at $t+1$ is certain. **Q.E.D.**

The link between Lemmas 3 and 4 is akin to the one between Lemmas 1 and 2. In Lemma 3, after a medium price, some types sell, while others—who observed the signal later—wait. Lemma 4 examines under what conditions the latter find it optimal to wait despite a sizable risk of getting caught in the crash. Again, the Lemma's proof is in Appendix B, but I will sketch the main argument here with the help of Figure 4. Consider type $n = t_0 + d + 2$ at $t = t_0 + \tau^{**} + 1.$ (Note that $t = t_0 + \tau^{**} + 1 = n + \tau^* - 1,$ i.e., type $n$ is the lowest among the types who stay in the market at $t$.) After $d+1$ high prices, type $t_0$ sold at $t-1,$ $p_{t-1}$ is medium, and now $d+1$ more types $t_0+1,\ldots,t_0+d+1$ will sell at $t,$ while types $n$ and higher wait. Clearly, type-$n$ agents would not wait if they knew $t_0.$ But given their information, they believe that there are $d+3$ possible values of $t_0,$ $n-(d+2),\ldots,n.$ They thus assign a probability $e^{-\lambda(d+2)}/(1+\cdots+e^{-\lambda(d+2)})$ to the possibility of being 'first in line', i.e., to $t_0 = n.$ Selling in this case would mean forgoing a sizable expected return $W_d.$ If $t_0 = n,$ for the next $d$ periods, with probability $1-\pi,$ type-$n$ agents will sell and with probability $\pi,$ the bubble will continue to grow. Overall, the return $W_d$ (see appendix B for full details) is given by

$$W_d = (1-\pi)\left(\frac{G}{R}\right)\frac{\left(\pi G/R\right)^{d+1}-1}{\pi G/R-1} + \left(\pi\frac{G}{R}\right)^{d+1}. \tag{26}$$

If $\pi G/R > e^{\lambda},$ as $d$ increases, $W_d$ grows faster than $e^{-\lambda(d+2)}/(1+\cdots+e^{-\lambda(d+2)})$ falls, and thus

$$1 < \frac{e^{-\lambda(d+2)}}{1+\cdots+e^{-\lambda(d+2)}}W_d \tag{27}$$

holds for large $d$. In other words, if $\pi G/R > e^{\lambda},$ there exist values of $d$ for which $W_d$ is so large that type-$n$ agents would be willing to wait at $t$, even if the price fell to zero for all $t_0 < n.$

In a second part of the proof, I show that, of all situations where the last price is medium and less than $\tau^*$ periods have passed since observing the signal, the one discussed above is the most critical one, in the sense that preemptive sales are most tempting. This is for two reasons. First, if there are less than $d+1$ consecutive high prices leading up to the medium $p_{t-1},$ type-$n$

agents can rule out some of the earlier values of $t_0$, which makes $t_0 = n$ relatively more likely. To see the second reason, consider type $n+1$ at time *t*. Just like for type *n*, for $n+1$, the support of $t_0$ at $t$ contains $d+2$ 'bad' values with $t_0 < n$. But for type $n+1$, the support has two 'good' values, $t_0 = n$ and $t_0 = n+1$, with expected return $W_d$ or higher. Type $n+1$ is thus less tempted to sell than type $n$. By the same token, types $n+2$ and higher are less tempted than type $n+1$.

**Lemma 4** — *Let $t < n+\tau^*$, and let $p_{t-1}$ be medium. If $\pi G/R > e^{\lambda}$ and $d+2 < \kappa N$, there is a threshold $\bar{d} > 0$ such that if $d \geq \bar{d}$, type-n agents find it optimal not to sell at time t.*

**Proof** — *See Appendix B.*

To recapitulate, in Proposition 2, I combine Lemmas 1-4 to obtain sufficient conditions to support bubbles in equilibrium.

***Proposition 2*** — *Suppose that $\pi \geq 1/(1+e^{-\lambda})$ and that $e^{\lambda}/\pi < G/R < \Gamma$. Then:*

*4.1 If $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi) \leq G/R$, there exists a threshold $\bar{d} \in \{1,2,3,...\}$, such that any $(\tau^*, \tau^{**})$ with $\tau^* \geq 0$, $\tau^{**} \geq 1$, $d \geq \bar{d}$ and $d+2 < \kappa N$ can be supported in equilibrium.*

*4.2 If $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi) > G/R$, there exists a threshold $\bar{d} \in \{1,2,3,...\}$, such that any $(\tau^{**}, \tau^*)$ with $\tau^* \geq 0, 1 \leq \tau^{**} \leq \ln\{\pi/[(1+e^{-\lambda})-(1+e^{-\lambda}-\pi)G/R]\}/\ln(G/R)-1$, $d \geq \bar{d}$ and $d+2 < \kappa N$ can be supported in equilibrium.*

***Proof*** — Start with (2.1). Since $\pi \geq 1/(1+e^{-\lambda})$, $d+2 < \kappa N$, $G/R < \Gamma$ and $\tau^* \geq 1$, by Lemma 1, type-*n* agents are willing to sell at $n+\tau^{**}$ with high $p_{n+\tau^{**}-1}$. And since $\pi \geq 1/(1+e^{-\lambda})$, $G/R < \Gamma$ $d+2 < \kappa N$, and $\tau^* \geq 0$, by Lemma 3, they are willing to sell at $t \geq n+\tau^* \geq 2$ with medium $p_{t-1}$. By Lemma 2, since $e^{\lambda}/\pi < G/R$, $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi) \leq G/R$ and $d+2 < \kappa N$, type-*n* agents are willing to wait at $t < n+\tau^{**}$ with high $p_{t-1}$. By Lemma 4, since $e^{\lambda}/\pi < G/R$, and $d+2 < \kappa N$, $\bar{d} > 0$ exists such that, if $d \geq \bar{d}$, type-*n* agents are willing to wait at $t < n+\tau^*$ with medium $p_{t-1}$.

To prove (2.2), invoke Lemmas 1, 3 and 4 exactly as before. Then, by Lemma 2, since $G/R < (1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$, type-*n* agents are willing not to sell at $t < n+\tau^{**}$ with high $p_{t-1}$ only for $\tau^{**} \leq \ln\{\pi/[(1+e^{-\lambda})-(1+e^{-\lambda}-\pi)G/R]\}/\ln(G/R)-1$. **Q.E.D.**

In sum, when noise cannot even hide sales by one type, Proposition 1 establishes that, if $e^{\lambda} < G/R < \Gamma,$ the only equilibrium (under strategic restrictions I-IV) has agents selling as soon as they observe the signal. And when there is enough noise to hide sales by one type, but not more, Proposition 2 establishes that arbitrarily long bubbles (with $d = \tau^{**} - \tau^{*},\ d + 2 < \kappa N,$ and $d \geq \overline{d}$ ) can be supported if $e^{\lambda}/\pi < G/R < \Gamma,\ \ \pi \geq 1/(1+e^{-\lambda})$ and $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi) < G/R.$ In Proposition 3, I show that all these conditions are compatible.[23]

***Proposition 3*** **—** *There exists a nonempty region of the parameter space where* $e^{\lambda} < G/R < \Gamma,$ $\pi \geq 1/(1+e^{-\lambda}),\quad (1+e^{-\lambda})/(1+e^{-\lambda}-\pi) < G/R$ *and* $e^{\lambda}/\pi < G/R.$

***Proof*** **—** In Figure 5, I plot all restrictions in a diagram with $\pi$ on the horizontal and $G/R$ on the vertical axis. The pairs $(\pi, G/R)$ of interest lie in the interior of the rectangle defined by $1/(1+e^{-\lambda}) \leq \pi \leq 1$ and $e^{\lambda} \leq G/R \leq \Gamma,$ and above the graphs of the functions $G/R = e^{\lambda}/\pi$ and $G/R = (1+e^{-\lambda})/(1+e^{-\lambda}-\pi).$ For $\pi \in [(1+e^{-\lambda})^{-1}, 1],$ both functions are continuous, $e^{\lambda}/\pi$ is strictly decreasing and $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$ strictly increasing in $\pi.$ They intersect at the point $(\pi_I, [G/R]_I)$, where $\pi_I = (1+e^{-\lambda})/(1+e^{-\lambda}+e^{-2\lambda})$ and $[G/R]_I = e^{\lambda}(1+e^{-\lambda}+e^{-2\lambda})/(1+e^{-\lambda}).$ For any $\lambda > 0,\ \ (\pi_I, [G/R]_I)$ is in the interior of the rectangle, as $\pi_I$ is clearly between $1/(1+e^{-\lambda})$ and 1, $[G/R]_I$ is clearly above $e^{\lambda},$ and, as I show in Appendix C, $[G/R]_I < \Gamma.$ Since $(\pi_I, [G/R]_I)$ is in the interior of the rectangle, there exists a region in the $(\pi, G/R)$ plane (the shaded area in Figure 5), where all parameter restrictions hold. **Q.E.D.**

For any $\lambda > 0,$ the region where all parameter restrictions are satisfied—the shaded area in Figure 5—is delimited by one straight side and two curved sides.[24]

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[23] Here I focus on the first part of Proposition 2 and ignore the second, by which, if $G/R < (1+e^{-\lambda})/(1+e^{-\lambda}-\pi),$ $\tau^{**}$ less than $\ln\{\pi/[(1+e^{-\lambda})-(1+e^{-\lambda}-\pi)G/R]\}/\ln(G/R)-1$ can be supported. Since the focus of the analysis is on long bubbles, and the upper bound on $\tau^{**}$ is small unless $(1+e^{-\lambda})-(1+e^{-\lambda}-\pi)G/R \approx 0,$ considering the case where $G/R < (1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$ in Proposition 3 would add more complication than insight.

[24] A few algrebra steps suffice to show that $e^{\lambda}/\Gamma > 1/(1+e^{-\lambda})$ and that $(1+e^{-\lambda})(\Gamma-1)/\Gamma < 1,$ where $e^{\lambda}/\Gamma$ and $(1+e^{-\lambda})(\Gamma-1)/\Gamma$ are, respectively, the values of $\pi$ for which the graphs of the functions $G/R = e^{\lambda}/\pi$ and $G/R = (1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$ cross the $G/R = \Gamma$ line.

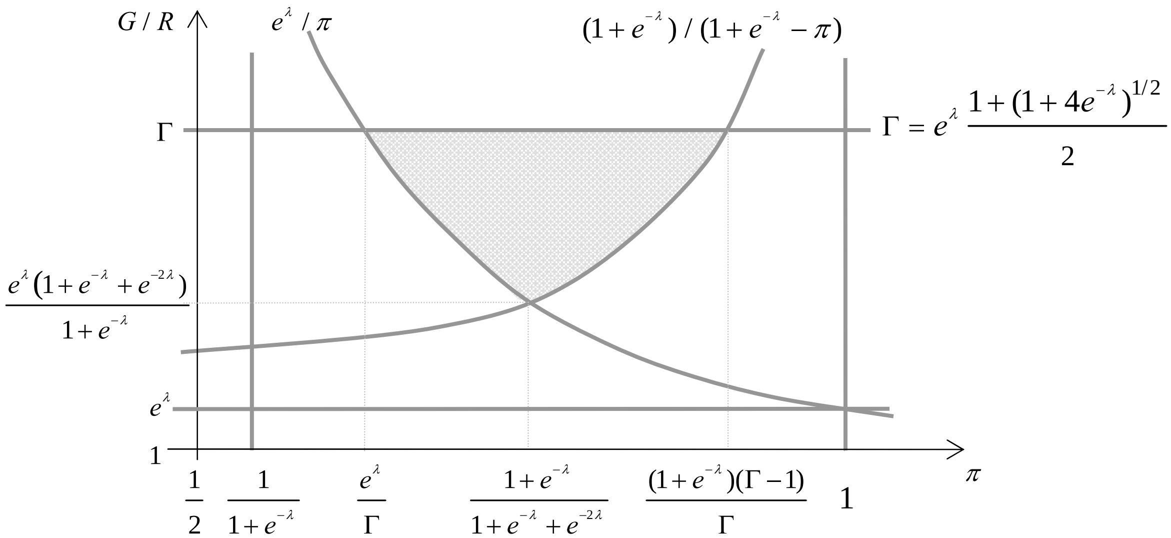


**Figure 5** — Compatibility of Parameter Restrictions from Propositions 1 and 2.

Having shown that there exist parameters for which no bubbles arise without noise, but long bubbles arise if noise can hide one sale, we can now construct bubbly equilibria as follows. Pick a pair $(\pi, G/R)$ inside the shaded area above.[25] Pick some large $\tau^{**}$, and let $\tau^{*} = \tau^{**} - \overline{d}$, where $\overline{d}$ is the smallest integer for which (27) holds. Finally, let $N$ be an integer above $(\overline{d}+2)/\kappa$, and set $\overline{\varepsilon} = 1/(2\pi N)$. More concretely, consider the following examples:

***__Numerical Example 1__*** **—** *Let* $\lambda = 0.001$, *implying* $\Gamma \approx 1.619$. *Set* $\pi = 0.75$ *and* $G/R = 1.618$. *Since* $G/R > (1+e^{-\lambda})/(1+e^{-\lambda}-\pi) \approx 1.6005$, $\pi G/R > e^{\lambda}$, *and* $G/R < \Gamma$, $(\pi, G/R)$ *lies inside the shaded area in Figure 5. Set (arbitrarily)* $\tau^{**} = 56$. *Since* (27) *holds for* $d \geq 6$, *let* $\tau^{*} = 50$. *Finally, if* $\kappa = 1/2$ *and* $N = 40$, $(6+2) < \kappa N$. *It is implied that* $\overline{\varepsilon} = 1/(2\pi N) = 1/60$.■

To illustrate this example in more detail, in Table 1, I track agents' beliefs and expected returns in the last few periods of a bubble with $(\tau^{*}, \tau^{**}) = (50, 56)$. To fix ideas, let $t_0 = 50$. As in Figure 4, I assume realizations of noise for which the bubble bursts as late as possible.

[25] Technically, there are no restrictions on $\lambda$. But intuition suggests that it should be relatively small. Specifically, it would seem implausible to have a large value of $N$ (representing very disperse opinions about $t_0$) unless the expected $t_0$, given by $1/(1-e^{-\lambda})$, was also relatively large.

Specifically, after a medium $p_{95}$, prices remain high for ten periods $96,\ldots,105$. Type 50 sells at time $t_0 + 56 = 106$, $p_{106}$ is medium, types $51,\ldots,57$ sell in period 107, $p_{107}$ is low, causing a crash at time 108.

***Numerical Example 2*** **—** *Maintain* $\lambda = 0.001$, $\pi = 0.75$, $N = 40$, $\kappa = 1/2$ *and* $\bar{\varepsilon} = 1/60$. *But let* $G/R = (1+e^{-0.001} - \nu)/(1+e^{-0.001} - \pi)$ *for some small* $\nu < 0$, *so that* $(\pi, G/R)$ *lies just below Figure 4's shaded area. Then, equilibria with* $(\tau^*, \tau^{**})$ *can be supported if* $\tau^* \geq 0$, $\tau^{**} \leq \ln\{0.75/\nu\}/\ln(G/R) - 1$, *and* $\tau^{**} - \tau^* \geq \bar{d} = 6$. *Long bubbles can be supported only if* $\nu$ *is very small. For instance, if* $\nu = 0.001$ $(\nu = 10^{-8})$, *it must be that* $\tau^{**} \leq 13$ $(\tau^{**} \leq 37)$. ■

# 5 Extension: Re-entering the Market after Selling

The no-reentry assumption does much to simplify the analysis, and is quite defensible in high-transaction-cost markets such as real estate. But in other asset markets, such as stock markets, transaction costs are low and frequent trading is typical. In these markets, the no-reentry assumption it is clearly unrealistic. Furthermore, it is not obvious that it is innocuous. It is thus worth asking whether bubbles continue to arise once agents are able to buy and sell whenever they want. In this subsection, I answer this question in the affirmative. Specifically, I show that there are bubbly equilibria where agents follow (22) even though they are allowed to reenter the market after selling.[26] In these bubbles, agents could sell and reenter, but choose not to. To examine whether the reentry option makes a difference, I will revisit Lemmas 1-4.

In situations captured by Lemmas 1 and 3, it is easy to see why agents choose not to reenter the market after selling. Along the equilibrium path, whenever an agent sells, she knows that the crash will arrive within one or, at most, two periods. Reentry would thus have to take place either at time $T$, i.e., when the price is about to collapse, or at time $T+1$, i.e., after the crash. Clearly, agents will not choose to reenter.

[26] The equilibrium concept from Section 3 should be expanded to account for the fact that, with allowed reentry, the payoff from selling is no longer given by the price. In the expanded definion (not presented here to conserve space, but available upon request), there is an expected return associated with being in the market and one associated with being out of the market, which reflects the reentry option. Every period, agents choose between the two returns. As in Section 3, these returns are well defined, since it is possible to work backwards from the post-crash payoffs.

| | | | | | | $t_0+\tau^*$ | | | | | | $t_0+\tau^{**}$ | $T$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Period** $t$ | **95** | **96** | **97** | **98** | **99** | **100** | **101** | **102** | **103** | **104** | **105** | **106** | **107** | **108** |
| **Price** $p_t$ | Medium | High | High | High | High | High | High | High | High | High | High | Medium | Low | $f_t$ |
| **Type** $n$ | | | | | | | | | | | | | | |
| **50** $=t_0$ | | | 47,…,50 | 47,…,50 | 47,…,50 | 47,…,50 | 47,…,50 | 47,…,50 | 47,…,50 | 48,49,50 | 49,50 | 50 | 50 | 50 |
| | | | | | | | | | 1.516 | 1.222 | **1.011** | 0.405 | 0 | 1 |
| **51** | $\text{supp}_{n,t}(t_0)$ | | 47,…,51 | 47,…,51 | 47,…,51 | 47,…,51 | 47,…,51 | 47,…,51 | 47,…,51 | 48,…,51 | 49,50,51 | 50,51 | 50,51 | 50 |
| | | | | | | | | | 1.876 | 1.516 | 1.222 | **1.011** | 0 | 1 |
| **52** | depends on prices before period 95. | | 47,…,52 | 47,…,52 | 47,…,52 | 47,…,52 | 47,…,52 | 47,…,52 | 47,…,52 | 48,…,52 | 49,…,52 | 50,51,52 | 50,…,52 | 50 |
| | | | | | | | | | 2.300 | 1.876 | 1.516 | 1.222 | 0 | 1 |
| **53** | | | 47,…,53 | 47,…,53 | 47,…,53 | 47,…,53 | 47,…,53 | 47,…,53 | 47,…,53 | 48,…,53 | 49,…,53 | 50,…,53 | 50,…,53 | 50 |
| | | | | | | | | | 2.796 | 2.300 | 1.876 | 1.516 | 0 | 1 |
| **54** | | | 47,…,54 | 47,…,54 | 47,…,54 | 47,…,54 | 47,…,54 | 47,…,54 | 47,…,54 | 48,…,54 | 49,…,54 | 50,…,54 | 50,…,54 | 50 |
| | | | | | | | | | 3.372 | 2.796 | 2.300 | 1.876 | 0 | 1 |
| **55** | | | 47,…,55 | 47,…,55 | 47,…,55 | 47,…,55 | 47,…,55 | 47,…,55 | 47,…,55 | 48,…,55 | 49,…,55 | 50,…,55 | 50,…,55 | 50 |
| | | | | | | | | | 4.052 | 3.372 | 2.796 | 2.300 | 0 | 1 |
| **56** | | | 47,…,56 | 47,…,56 | 47,…,56 | 47,…,56 | 47,…,56 | 47,…,56 | 47,…,56 | 48,…,56 | 49,…,56 | 50,…,56 | 50,…,56 | 50 |
| | | | | | | | | | 4.828 | 4.052 | 3.372 | 2.796 | 0 | 1 |
| **57** | | | 47,…,57 | 47,…,57 | 47,…,57 | 47,…,57 | 47,…,57 | 47,…,57 | 47,…,57 | 48,…,57 | 49,…,57 | 50,…,57 | 50,…,57 | 50 |
| | | | | | | | | | 5.728 | 4.828 | 4.052 | 3.372 | 0.050 | 1 |
| **58** | | | 47,…,58 | 47,…,58 | 47,…,58 | 47,…,58 | 47,…,58 | 47,…,58 | 47,…,58 | 48,…,58 | 49,…,58 | 50,…,58 | 50,…,58 | 50 |
| | | | | | | | | | 6.773 | 5.728 | 4.828 | 4.052 | **1.032** | 1 |

**Table 1 —** Numerical Example 1: $\lambda = 0.001$, $\pi = 0.75$, $G/R = 1.618$, $\kappa = 1/2$, $N = 40$, $\overline{\varepsilon} = 1/60$. Equilibrium $(\tau^*, \tau^{**}) = (50, 56)$. To fix ideas, let $t_0 = 50$. As in Figure 4, the realizations of noise are such that the bubble bursts as late as possible. If a cell has no shading, the type is still in the market. Light shading denotes that the type is currently selling, and darker shading that the type already sold in a previous period. Each cell contains, at the top, $\text{supp}_{n,t}(t_0)$. When agents see a high $p_{96}$, they rule out $t_0 \leq 46$. High prices in periods 97-102 reveal no new information. Only as agents see high $p_{103}$, $p_{104}$ and $p_{105}$, they can successively discard 47, 48 and 49 from $\text{supp}_{n,t}(t_0)$. From period $t = 103$ onward, each cell also reports, at the bottom, the expected return from waiting relative to selling. If this is above (below) one, agents wait (sell). Post-crash, agents are indifferent between buying, selling and doing nothing. The boldfaced 1.011 for type 50 at time 105 and 51 at 106 is the right hand side of (24) evaluated at these parameter values. As we move down/left from these cells, the payoffs associated with waiting increase, as the probability of a crash at $t+1$ falls. The boldfaced 1.032 for type 58 at time 107 equals the right-hand-side of (27). As I show in the proofs of Lemmas 2 and 4, these cells represent the situations where types who are supposed to wait are most tempted to sell preemptively. In periods 97-102, agents know for sure that nobody is selling, and thus, have no incentive to sell. Details on how to calculate payoffs for all cells are available upon request. (As noted in footnote 8, the assumption that $\text{supp}_{n,108}(t_0) = \{50\}$ for all $n$ is literally true if $p_{107} < G^{107} - \alpha[\overline{\varepsilon} - 7/N]$, and approximately true otherwise. That is, if $p_{107} > G^{107} - \alpha[\overline{\varepsilon} - 7/N]$, type-50 agents have $\text{supp}_{50,108}(t_0) = \{50\}$, while others have $\text{supp}_{n,108}(t_0) = \{50, 51\}$.)

In situations captured by Lemmas 1 and 3, it is easy to see why agents choose not to reenter the market after selling. Along the equilibrium path, whenever an agent sells, she knows that the crash will arrive within one or, at most, two periods. Reentry would thus have to take place either at time $T$, i.e., when the price is about to collapse, or at time $T+1$, i.e., after the crash. Clearly, agents will not choose to reenter.

The reentry option turns out not to make a difference in situations captured by Lemma 2, either. Roughly (see Appendix D for details), reentry does not matter for the following reason. The reentry option makes preemptive sales more desirable by reducing their potential opportunity cost. After selling, if the bubble does not burst, reentering agents forego just one, instead of many, periods of appreciation. However, as I show in the proof of Lemma 2, of all sitations with $t < n + \tau^{**}$ and high $p_{t-1}$, the most critical situation for type-*n* agents is the one captured by (24). In this situation, even if the bubble does not burst at $t$, type-*n* agents will sell at time $t+1$. This means that, even with forbidden reentry, the relevant opportunity cost to rule out preemptive sales already consists of just one period of forgone profit. Since the reentry option cannot reduce this opportunity cost, it cannot tilt the balance in favor of preemptive selling.

Given the above reasoning, it is not surprising that the situations covered by Lemma 4 are the ones where the reentry option makes the greatest difference. In Lemma 4, if $\pi G / R > e^{\lambda}$ and $d \geq \bar{d}$, type-*n* agents wait at $t = n + \tau^{*} - 1$ with medium $p_{t-1}$ (preceded by $d+1$ high prices) for fear of missing out on a large return $W_d$ if $t_0 = n$. This return is accumulated over a number of periods that may reach up to $d+1$ periods. Allowing reentry lowers this opportunity cost, since an agent who sells at $t$ and then sees a high $p_t$ can reenter at $t+1$, foregoing only part of $W_d$. The sell-or-wait choice is no longer governed by (27), since the return from selling (on the left) now includes a reentry return $W_{d-1}$ with probability $\pi e^{-\lambda(d+2)} / (1 + \cdots + e^{-\lambda(d+2)})$—the probability that $t_0 = n$ and $p_t$ is high. Hence, waiting is now optimal if

$$1 + \pi \frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}} \left(W_{d-1} - 1\right) < \frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}} W_d.$$

Since $W_{d-1} > 1$ for all $d \geq 1$, agents prefer selling and reentering (if $p_t$ is high) to selling without the option to reenter. Still, if $d$—and hence $W_d$—is large enough, not selling is better than selling and reentering. This is because agents who sell and reenter forego one period of growth, and thus, part of $W_d$. For large enough $d$, this foregone part—which is related to the difference between $W_d$ and $W_{d-1}$—is important enough to deter preemptive sales. To see this more precisely, use (26) and rearrange terms to rewrite the above inequality as

$$\frac{1-e^{-\lambda(d+3)}}{1-e^{-\lambda}}-e^{-\lambda(d+2)}\left[\pi+\frac{G/R(1-\pi)^2}{\pi G/R-1}\right]<e^{-\lambda(d+2)}\left(\pi\frac{G}{R}\right)^d\frac{\pi(G/R-1)^2}{\pi G/R-1}. \quad (28)$$

Note that the left hand side increases with $d$, but approaches $1/(1-e^{-\lambda})$ as $d\to\infty$. The right hand side, since $\pi G/R>e^{\lambda}$, grows exponentially with $d$. Thus, there is a positive $\bar{\bar{d}}>\bar{d}$ such that (28) holds if $d\geq\bar{\bar{d}}$. However, once reentry is allowed, equilibria with $\bar{d}\leq d<\bar{\bar{d}}$ vanish.[28]

Other than the fact that the minimum $d$ is lengthened from $\bar{d}$ to $\bar{\bar{d}}$, there are no new requirements that equilibria with bubbles must satisfy once reentry is allowed. Hence, within the class of equilibria where agents follow (22), the possibility of reentry makes a quantitative, but not a qualitative, difference. The mechanisms protecting bubbles from preemptive sales remain the same, and long bubbles still arise. For parameter values as in *Numerical Example* 1, $\bar{\bar{d}}=13$, and thus, pairs $(\tau^*,\tau^{**})=(k,k+d)$, with $k\geq 0$ and $13\leq d<18$ satisfy all inequalities. However, equilibria from *Example 1* with $6\leq d<12$ vanish, as they are not "reentry proof".

# 6 Conclusion

This paper extends existing models of greater fool's bubbles (Allen, et al. (1993), Conlon (2004), and especially Abreu and Brunnermeier (2003)) by considering a fully rational economy with noisy prices and price responsiveness to selling pressure. The fact that bubbles arise in such an environment shows that it is possible to model the notion of information-driven speculation in a robust way, without sacrificing rationality, common priors, or market-clearing prices. By bringing models of speculation one step closer to standard economic theory, this paper contributes towards the goal of developing models that may be useful to address questions of optimal policy in the presence of bubbles.

For future work, in addition to extending the theory to analyze policy, it may be desirable to refine some aspects of the model, for example by modeling the source of the growing

[28] Inequality (28) dissuades type-$n$ agents from selling at $t=n+\tau^*-j$ with medium $p_{t-1}$ also if $j>1$ or less than $d+1$ high prices precede $p_{t-1}$. As in Lemma 4, both of these changes make preemptive selling less tempting.

availability of resources that can be invested into the bubble. This could perhaps be done by modeling leverage, or by modeling late arrival of agents into the market. Moreover, as Brunnermeier (2001) points out, in models bubbles burst typically burst abruptly, while in reality, they often deflate gradually. A version of the current model where the noisy component was not bounded might generate this sort of gradual decline.

**APPENDIX A — Derivation of $\Gamma$**

Evaluate (20) at $\tau^*=1$, set $x=G/R$ and find roots of $1=x^{-2}/(1+e^{-\lambda})+xe^{-\lambda}/(1+e^{-\lambda})$:

$1+e^{-\lambda}=x^{-2}+e^{-\lambda}x \Leftrightarrow (1+e^{-\lambda})x^2=1+e^{-\lambda}x^3 \Leftrightarrow x^2-1=e^{-\lambda}(x^3-x^2) \Leftrightarrow (x+1)(x-1)=e^{-\lambda}x^2(x-1).$ Clearly, $x=1$ is a root. For $x\neq 1,$ we have, $x+1=e^{-\lambda}x^2 \Leftrightarrow 0=x^2-e^{\lambda}x-e^{\lambda}.$ The quadratic formula yields roots $x=(1\pm\sqrt{1+4e^{-\lambda}})e^{\lambda}/2.$ Let $\Gamma=e^{\lambda}(1+\sqrt{1+4e^{-\lambda}})/2$ be the positive root, which is always above one. ($\Gamma$ is increasing in $\lambda$ and approaches $(1+\sqrt{5})/2\approx 1.618$ as $\lambda\to 0.$ ) Inequality (20) evaluated at $\tau^*=1$ holds if $1<G/R<\Gamma$ and fails if $G/R\geq\Gamma.$

**APPENDIX B — Proofs of Lemmas 2 and 4**

**Proof of Lemma 2 —** Consider an arbitary type $n\geq 1$ at $t=n+\tau^{**}-j,$ for any $\tau^{**}\geq 0$ and $j\geq 1.$ Let $p_{t-1}$ be high. A type-*n* agent may be inclined to sell preemptively at $t$ for fear that the bubble may burst at $t+1.$ In fact, if $t_0=n-j,$ type-$t_0$ agents will sell at $t=n+\tau^{**}-j,$ causing a crash at $t+1$ with probability $\pi.$ In, and only in, the following cases *(i)*-*(iv)*, type-*n* agents are not tempted to sell at $t$ because $t_0=n-j$ is either impossible *(i-iii)*, or very unlikely *(iv)*:

- *(i)* If at least one of the prices $p_{t-(d+1)},\ldots,p_{t-2}$ is medium, $t_0$ cannot be $n-j.$ (Note that, since $t-(d+1)=n-j+\tau^*-1,$ if $p_{t-s}$ was medium for some $s\in\{2,\ldots,d+1\}$ and $t_0$ was $n-j,$ type-$t_0$ agents would have sold at time $t-s+1,$ and $p_{t-1}$ would not be high.)
- *(ii)* If $j\geq n,$ $t_0$ cannot be $n-j,$ since $t_0$ cannot be less than 1.
- *(iii)* If $j\geq N,$ $t_0$ cannot be $n-j,$ since $t_0$ cannot be less than $n-(N-1).$
- *(iv)* If $j>\tau^{**},$ type-*n* agents have yet to observe the signal as of time $t.$ If $\tau^{**}\geq N-1,$ sales cannot begin before all signals arrive. If $\tau^{**}<N-1,$ $\mathrm{supp}_{n,t}(t_0)=\{\tau_0\mid\tau_0\geq n-j\},$ and $\mu_{n,t}(n-j)$ is $1-e^{-\lambda}.$ Since $e^{\lambda}<G/R,$ type-*n* agents prefer not to sell at $t.$

Having ruled out preemptive sales if one or more of *(i)*-*(iv)* hold, it remains to discuss situations where none of these conditions apply, i.e., cases where $j\leq\min\{\tau^{**},n-1,N-1\}$ and $p_{t-s}$ is high $\forall s=1,\ldots,d+1.$ To rule out preemptive sales in these cases, it suffices to focus on the case where $j=1.$ To see why, note that at $t=n+\tau^{**}-j,$ $\mathrm{supp}_{n,t}(t_0)=\{n-j,\ldots,n\},$ with the probability that $t_0=n-j$ given by $\mu_{n,t}(n-j)=1/(1+e^{-\lambda}+\cdots+e^{-\lambda j}),$ which is greatest for $j=1.$ Thus, if type-*n* agents do not to sell preemptively if $j=1,$ they will not do so either if

$j > 1.$ Let us then consider the situation faced by type-*n* agents at $t = n + \tau^{**} - 1.$ High prices $p_{t-(d+1)}, \dots, p_{t-1}$ reveal to type-*n* agents that they were either first or second to observe the signal, i.e., that $t_0$ must be $n-1$ or $n.$ By (), probabilities $\mu_{n,t}(n-1)$ and $\mu_{n,t}(n),$ are respectively given by $1/(1+e^{-\lambda})$ and $e^{-\lambda}/(1+e^{-\lambda}).$ If $t_0 = n-1,$ the first type will sell at $t.$ With probability $\pi,$ $p_t$ will be low, causing a crash at $t+1,$ and with probability $1-\pi,$ $p_t$ will be medium, and $d+1$ types will sell at $t+1$ at the expected price $G^{t+1}.$ (Since $d+2 < \kappa N,$ behavioral agents will be able to buy the shares). If $t_0 = n,$ nobody will sell at $t,$ and type-*n* agents will sell at $t+1$ at a price $G^{t+1}.$ In sum, waiting is best if (24) holds. If $G/R \geq (1+e^{-\lambda})/(1+e^{-\lambda}-\pi),$ (24) holds for any $\tau^{**}.$ Otherwise, it holds only for $\tau^{**}$ under the threshold given by (25). **Q.E.D.**

**<u>Proof of Lemma 4</u> —** The proof proceeds in two steps. In *Step 1*, I derive conditions under which type-*n* agents choose not to sell at $t = n + \tau^* - 1$ with medium $p_{t-1}$ if $n \geq d+3,$ $\tau^* \geq 1,$ and $p_{t-s}$ is high $\forall s = 2, \dots, d+2$. In *Step 2*, I show that, of all possible situations cases with $t < n + \tau^*$ and medium $p_{t-1},$ type-*n* agents are most tempted to sell in the situation considered in *Step 1*. Consequently, the conditions ruling out preemptive sales in *Step 1* also suffice to rule out preemptive sales by type-*n* agents in all other situations with $t < n + \tau^*$ and medium $p_{t-1}.$

*<u>Step 1</u>* — Suppose that $t = n + \tau^* - 1,$ $p_{t-1}$ is medium, $n \geq d+3,$ $\tau^* \geq 1,$ and $p_{t-s}$ is high $\forall s = 2, \dots, d+2.$ Then, type-*n* agents think that $t_0$ could be anywhere from $n-(d+2)$ to $n,$ i.e., $\text{supp}_{n,t}(t_0)$ is $\{n-(d+2), \dots, n\}.$ If $n-(d+2) \leq t_0 \leq n-2,$ two or more types will sell at $t,$ causing a crash. If $t_0 = n-1,$ type $n-1$ will sell, causing a crash with probability $\pi.$ In sum, if $t_0 < n,$ waiting brings losses. To simplify, assume that, if $t_0 < n,$ the payoff from waiting is zero. Given this, type-*n* agents will wait at $t$ only if the expected (gross) return $W_d$ if $t_0 = n$ is sufficiently large. Since $e^{-\lambda(d+2)}/(1+\dots+e^{-\lambda(d+2)})$ is the probability that $t_0 = n,$ (27) is a sufficient condition for type-*n* agents to be willing to wait at time $t.$

To derive (26), note that $W_d$ depends on how long type-*n* agents ride the bubble after $t,$ which could amount to a maximum of $d+1$ periods. If $t_0 = n,$ every period from $t+1 = n + \tau^*$ to $t+d = n + \tau^{**} - 1,$ type-*n* agents will sell if the last price is medium (which will occur with probability $1-\pi)$ and wait if it is high (which will occur with probability $\pi).$ If all prices

$p_t, \ldots, p_{t+d-1}$ are high, they will sell at $t+d+1 = n+\tau^{**}$, regardless of whether $p_{t+d}$ is high or medium. Thus, $W_d$ is given by

$$W_d = (1-\pi)\frac{G}{R}\left(1+\pi\frac{G}{R}+\left(\pi\frac{G}{R}\right)^2+\cdots+\left(\pi\frac{G}{R}\right)^d\right)+\left(\pi\frac{G}{R}\right)^{d+1}.$$

Since $\pi G/R \neq 1$, this can be rewritten as (26). Since $\pi G/R > e^{\lambda}$, $e^{-\lambda(d+2)}W_d$ grows exponentially with $d$. In turn, this implies that (27) holds for high enough $d$. To see this, substitute (26) into (27) and rearrange terms to obtain

$$\frac{1-e^{-\lambda(d+3)}}{1-e^{-\lambda}} < e^{-\lambda(d+2)}\left\{(1-\pi)\left(\frac{G}{R}\right)\frac{(\pi G/R)^{d+1}-1}{\pi G/R-1}+\left(\pi\frac{G}{R}\right)^{d+1}\right\}. \tag{29}$$

The left-hand-side of (29) is increasing in $d$, but it approaches $1/(1-e^{-\lambda})$ as $d \to \infty$. On the right, all terms are positive, and since $\pi G/R > e^{\lambda}$, some terms grow exponentially with $d$. Thus, (29) holds for $d$ above some threshold $\bar{d}$. Finally, I derived (26) and (29) assuming that all is lost in the crash, a good approximation if $\tau^*$ is large. But for small $\tau^*$, type-*n* agents are even less inclined to sell at $t$, because they will lose less in the event of a crash.

_Step 2_ — Of all possible cases with $t < n+\tau^*$ and medium $p_{t-1}$, type-*n* agents are most inclined to sell if $t = n+\tau^*-1$, $\tau^* \geq 1$, and $p_{t-s}$ is high $\forall s \in \{2,\ldots,d+2\}$. In this case, the support of $t_0$ contains $d+2$ 'bad' values $n-(d+2),\ldots,n-1$, for which $t_0 < n$, and one 'good' value, $t_0 = n$, for which waiting at $t$ yields a large return $W_d$. In all other cases with $t < n+\tau^*$ and medium $p_{t-1}$ the support of $t_0$ contains less bad values and/or more good values, making a crash at $t+1$ less likely, and a large return more likely. To see this, observe how $\text{supp}_{n,t}(t_0)$ changes when $p_{t-1}$ is medium, but it is no longer the case that $t = n+\tau^*-1$, $\tau^* \geq 1$, $n \geq d+3$, and $p_{t-s}$ is high $\forall s \in \{2,\ldots,d+2\}$. Every change in conditions makes waiting more attractive, by reducing the number of bad values or increasing the number of good values of $t_0$ in $\text{supp}_{n,t}(t_0)$.

*(i)* Let $t = n+\tau^*-j$, $j \geq 2$, and $j \leq \tau^*$, (with $n \geq d+3$ and high $p_{t-s}$ $\forall s \in \{2,\ldots,d+2\}$). If $j \leq N-(d+2)$, the set $\text{supp}_{n,t}(t_0) = \{n-j-(d+1),\ldots,n\}$ contains $d+2$ bad values $n-j-(d+1),\ldots,n-j$, and $j$ good values $n-j+1,\ldots,n$. Moreover, the expected return

if $t_0 > n-j+1$ exceeds $W_d$. If $j > N-(d+2)$, there are less than $d+2$ bad values values, because $n-j-(d+1) < n-(N-1)$.

*(ii)* If $t = n+\tau^*-j$ and $j > \tau^*$, (with $n \geq d+3$ and high $p_{t-s}$ $\forall s \in \{2,\ldots,d+2\}$), $\text{supp}_{n,t}(t_0)$ equals $\{\tau \mid \tau \geq n-j-(d+1)\}$, i.e., type-*n* agents have yet to observe the signal as of time $t$. There may be up to $d+2$ bad values, but there are infinitely many good values.

*(iii)* If $n < d+3$, (with $t = n+\tau^*-1$, $\tau^* \geq 1$, and high $p_{t-s}$ $\forall s \in \{2,\ldots,d+2\}$), type-*n* agents know, from their signal, that $t_0$ cannot be $n-(d+2)$, since $n-(d+2) \leq 0$. Type-*n* agents can thus eliminate $d+3-n$ bad values from $\text{supp}_{n,t}(t_0)$.

*(iv)* If there are $k < d+1$ consecutive high prices before $p_{t-1}$, $\text{supp}_{n,t}(t_0) = \{n-(k+1),\ldots,n\}$, i.e., the number of bad values falls from $d+2$ to $k+1$. This makes the good value $t_0 = n$ relatively more likely. Also note that $p_{t-s}$ can be high $\forall s \in \{2,\ldots,d+2\}$ only if $t \geq d+3$.

If more than one of *(i)-(iv)* apply, several factors make waiting more desirable than in Step 1. The number of good values exceeds 1 and/or the number of bad values falls below $d+2$. **Q.E.D.**

## <u>APPENDIX C</u> — *Proof that* $[G/R]_I < \Gamma$.

$$[G/R]_I < \Gamma \Leftrightarrow \frac{\cancel{e^{\lambda}}(1+e^{-\lambda}+e^{-2\lambda})}{1+e^{-\lambda}} < \frac{\cancel{e^{\lambda}}(1+\sqrt{1+4e^{-\lambda}})}{2} \Leftrightarrow 2\left(1+\frac{e^{-2\lambda}}{1+e^{-\lambda}}\right) < 1+\sqrt{1+4e^{-\lambda}}$$

$$\Leftrightarrow 1+\frac{2e^{-2\lambda}}{1+e^{-\lambda}} < \sqrt{1+4e^{-\lambda}} \Leftrightarrow \cancel{1}+\frac{4e^{-2\lambda}}{1+e^{-\lambda}}+\frac{4e^{-4\lambda}}{(1+e^{-\lambda})^2} < \cancel{1}+4e^{-\lambda} \Leftrightarrow \frac{\cancel{4}e^{-2\lambda}}{1+e^{-\lambda}}+\frac{\cancel{4}e^{-4\lambda}}{(1+e^{-\lambda})^2} < \cancel{4}e^{-\lambda}$$

$$\Leftrightarrow e^{-2\lambda}(1+e^{-\lambda})+e^{-4\lambda} < e^{-\lambda}(1+e^{-\lambda})^2 \Leftrightarrow e^{-\lambda}+e^{-2\lambda}+e^{-3\lambda} < 1+2e^{-\lambda}+e^{-2\lambda} \Leftrightarrow e^{-3\lambda} < 1+e^{-\lambda}.$$ **Q.E.D.**

## <u>APPENDIX D</u> — *Details of Subsection 5.1*

Consider a type-*n* agent at time $t = n+\tau^{**}-j$, with $j \geq 1$ and high $p_{t-1}$. Instead of waiting, agents can sell with the option to reenter later. This has the benefit of protecting the agent against a crash if the bubble bursts, and the cost of foregoing capital gains (between the time of sale and reentry) if the bubble continues to grow. Clearly, since the benefit from preemptive selling is avoiding a crash, preemptive selling is suboptimal in cases *(i)-(iv)*, as listed in the proof of Lemma 2. In all these cases, the crash probability is either zero or very small.

Thus, as in Lemma 2, the situations of interest are those with $j \leq \min\{\tau^{**}, n-1, N-1\}$ and high $p_{t-s}$ $\forall s = 1,\ldots,d+1$. To revisit type-*n* agents' sell-or-wait tradeoffs in these situations, let us first invest a little in notation, letting $\psi_j$ denote the (gross expected discounted) return earned by a type-$n$ agent if she follows the equilibrium strategy from $t = n + \tau^{**} - j$ onward. Note that $\psi_1$ equals the right-hand-side of (24), and that, if $2 \le j \le d+1$,

$$\psi_j = \pi \frac{1}{1+\cdots+e^{-\lambda j}}\left(\frac{G}{R}\right)^{-(\tau^{**}+1)} + \frac{G}{R}\left[(1-\pi)+\pi\left(1-\frac{1}{1+\cdots+e^{-\lambda j}}\right)\psi_{j-1}\right].$$

That is, if the agent waits at $t$, the bubble will burst with probability $\pi/(1+\cdots+e^{-\lambda j})$, the agent will sell at $t+1$ if $p_t$ is medium—which will happen with probability $(1-\pi)$—and with the remaining probability, $p_t$ will be high and the agent will earn $G/R$ times $\psi_{j-1}$. Now compare this with the expected return from selling preemptively and reentering at $t+1$ if $p_t$ is high. This return is 1 if the agent does not reenter and $\psi_{j-1}$ if she reenters—which she will do with probability $(1-\pi)$. Hence, the return from selling and possibly reentering is

$$1-\pi\left(1-\frac{1}{1+\cdots+e^{-\lambda j}}\right)+\pi\left(1-\frac{1}{1+\cdots+e^{-\lambda j}}\right)\psi_{j-1}.$$

Clearly, if (24) holds and $\psi_{j-1} = 1$, $\psi_j$ exceeds the value of this last expression. The difference only grows when taking into account the fact that, by Lemma 2, $\psi_{j-1} > 1$ for $2 \le j \le d+1$. A similar argument (a bit more cumbersome notationally, and available upon request) rules out preemptive selling with reentry option if $j$ exceeds $d+1$, in which case agents will reenter the market after selling even after a medium $p_t$.

Finally, note that, since staying in the market is better than selling preemptively (with reentry option) for all $j$, agents have no incentive to sell preemptively and reenter after multiple periods. Staying out for more than one period serves only to compound the expected opportunity costs relative to staying in the market.

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